Chaos, Solitons and Fractals 85 (2016) 68–76

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Joint survival probability via truncated invariant copula

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ARTICLE INFO

Article history:
Received 21 August 2015
Revised 23 December 2015
Accepted 20 January 2016
Available online 12 February 2016

Keywords:
Joint survival probability
Truncated invariant FGM copula
Shot noise process
Basket default swap
Intensity model

ABSTRACT

Given an intensity-based credit risk model, this paper studies dependence structure between default intensities. To model this structure, we use a multivariate shot noise intensity process, where jumps occur simultaneously and their sizes are correlated. Through very lengthy algebra, we obtain explicitly the joint survival probability of the integrated intensities by using the truncated invariant Farlie–Gumbel–Morgenstern copula with exponential marginal distributions. We also apply our theoretical result to pricing basket default swap spreads. This result can provide a useful guide for credit risk management.

1. Introduction

The credit risk modeling has been developed in two ways: the structural approach and the intensity-based approach. Our current paper concerns the latter. Managing credit risks is challenging because of a disproportionately large amount of joint defaults by different counterparties. Joint default events impact financial markets significantly; they also create complicated mathematical problems. One of the best ways to understand the dependence structure of joint defaults is by using copulas. A copula is a function that links univariate distributions on $[0, 1]^n$ to a joint multivariate distribution on [0, 1] and provides one method of modeling the dependence structure. The existence of such a function is guaranteed by the well-known Sklar theorem (See Nelsen [12] for more details). Also, refer to Schweizer and Wolff [14] for the basic properties of copulas, Sungur [15] for the utilization of copulas to establish dependence models for multivariate data, and Choe et al. [2] developed a theoretical framework addressing the joint distribution and provided a general equation for time-dependent copulas related to stochastic processes that arise in finance.

A copula approach has been developed to deal with default correlations. Li [10] introduced the one-factor Gaussian copula and showed how to build a multivariate distribution of survival times for the valuation of credit derivatives. To relax the assumption of the Gaussian distribution in the one-factor Gaussian copula, student t-copula [5,6] and Clayton copula [13] are commonly used. Also, as shown by Ma and Kim [11], the copula approach enables modeling of default correlation under an intensity-based framework, where the default intensities are driven by dependent jumps.

This study is basically a combination of Ma and Kim [11] and Kim et al. [8]. Ma and Kim [11] derived a theoretical result for the joint survival probability under the Farlie–Gumbel–Morgenstern (FGM) copula in the credit default swap contract. Kim et al. [8] also proposed the generalized FGM copula and can be resulted using foreign currency exchange data. In financial market, the default events can be affected by the external events such as the bankruptcy of firms and mergers and acquisitions. To capture the ef-

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http://dx.doi.org/10.1016/j.chaos.2016.01.012

fects of external events, we consider that each entity has its own default intensity process and dependency structure between default intensities is modeled by the copula function. Generally, these external events can be resulted in a simultaneous positive jump in intensity processes. As time goes by, these intensity processes decrease to prevent default events after the arrival of a external event; the decreases continuous until another external event occurs. To model this situation, we consider a multivariate shot noise intensity process within Cox processes. So, this paper is based on correlated default intensities and is emphasized the impact of correlated jumps in intensities. We obtain joint Laplace transforms of the integrated intensities that provides us with the joint survival probability. It is understood numerically for the functional behavior of the joint survival probability with respect to the dependence parameters of the truncated invariant copula. To understand the modeling motivation, we apply the joint survival probability to pricing the basket default swap (BDS) rate. A BDS provides one party (the protection buyer) with a kind of insurance against the possibility of default in exchange for fixed premium to another (the protection seller) for protection related to credit events on an underlying portfolio of corporate bonds or other assets (reference credits) subject to default. Particularly, we suppose that each reference credits has a default intensity process and there exist jumps in the default intensity processes when an underlying portfolio of corporate bonds or other assets subject to default has simultaneous jumps. To make some simple calculations on the BDS price, the underlying portfolio is assumed to be a homogeneous group, which is equivalent to saying (mathematically) exchangeable (refer to Definition 2 in Section 5). Our main focus is a study of the dependence structure of the underlying portfolio.

The remainder of this paper is structured as follows. In Section 2, we describe one of the popular copulas, that is, the truncated invariant FGM copula. In Section 3, we calculate the joint Laplace transforms of the shot noise intensities using the FGM copula. In Section 4, we numerically investigate the functional behavior of the joint survival probability with respect to the dependence parameters of the FGM copula. In Section 5, we price the BDS rate under an intensity-based model for a homogeneous portfolio. We present our concluding remarks in Section 6.

2. Review of the truncated invariant FGM copula

In this section, we study a special class of n-dimensional distributions that could be uniquely determined by 2-dimensional marginal distributions developed by Sungur [16]. His idea is that 2-dimensional marginals determine the 3-dimensional copula under the truncation dependence invariance on the third variable. In this way, n-dimensional copula can readily be produced by using 2-dimensional one.

For each $i = 1, 2, 3,$ let the random variable X^i have a marginal distribution function denoted by $F_{X^i}(x^i)$. The joint distribution function and the related copula are denoted by $F_{X^1X^2X^3}(x^1, x^2, x^3)$ and $C_{X^1X^2X^3}(F_{X^1}(x^1), F_{X^2}(x^2), F_{X^3}(x^3))$, respectively. Let $T_{X^i} = \{x^i : x^i > a^i\}$ be the right-sided marginal truncation region for X^i. Then, the joint distribution function of $F_{X^1X^2X^3}(x^1, x^2, x^3)$ can be expressed as

$$F_{X^1X^2X^3}(x^1, x^2, x^3) = C_{X^1X^2X^3}(F_{X^1}(x^1), F_{X^2}(x^2), F_{X^3}(x^3))$$
$$= P\left(\bigcap_{m=1, m\neq i}^{3} \{X^m \le x^m\} \middle| \{X^i \le x^i\}\right) F_{X^i}(x^i).$$

Now, if $X_{tr}^{j(i)}$ and $X_{tr}^{k(i)}$ represent truncation random variables over the truncated region T_{X^i}, with the joint distribution function $F_{X_{tr}^{j(i)}X_{tr}^{k(i)}}$ and marginal distribution functions $F_{X_{tr}^{j(i)}}$ and $F_{X_{tr}^{k(i)}}$, then we have

$$\begin{aligned} F_{X_{tr}^{j(i)}X_{tr}^{k(i)}}(x^j, x^k) &= \frac{P(X^j \le x^j, X^k \le x^k, X^i \le a^i)}{P(X^i \le a^i)} \\ &= \frac{C_{a^i}(F_{X_{tr}^{j(i)}}(x^j), F_{X_{tr}^{k(i)}}(x^k))}{F_{X^i}(a^i)} \\ &= \frac{C_{a^i}\left(\frac{F_{X^jX^i}(x^j, a^i)}{F_{X^i}(a^i)}, \frac{F_{X^kX^i}(x^k, a^i)}{F_{X^i}(a^i)}\right)}{F_{X^i}(a^i)} \\ &= \frac{C_{a^i}\left(\frac{C_{X^jX^i}(F_{X^j}(x^j), F_{X^i}(a^i))}{F_{X^i}(a^i)}, \frac{C_{X^kX^i}(F_{X^k}(x^k), F_{X^i}(a^i))}{F_{X^i}(a^i)}\right)}{F_{X^i}(a^i)}, \end{aligned}$$

where C_{a^i} is the copula related to $X_{tr}^{j(i)}$ and $X_{tr}^{k(i)}$ as a function of the right truncation point a^i. Based on the above observation, we have the following result.

Theorem 1. *Let (X^1, X^2, X^3) be a random vector with a copula $C_{X^1X^2X^3}(u^1, u^2, u^3)$, where u^1, u^2, and u^3 are marginal distributions. Then, $C_{X_{tr}^{j(i)}X_{tr}^{k(i)}}(u^j, u^k)$ (the dependence structure of the random pair (X^j, X^k) over the truncated region) is dependent on a^i if and only if $C_{X^1X^2X^3}(u^1, u^2, u^3)$ can be represented as*

$$C_{X^1X^2X^3}(u^1, u^2, u^3) = C_{X^jX^k}\left(\frac{C_{X^jX^i}(u^j, u^i)}{u^i}, \frac{C_{X^kX^i}(u^k, u^i)}{u^i}\right)u^i,$$

where $i \neq j, j \neq k, k \neq i \in \{1, 2, 3\}$.

Proof. For the proof of this theorem, please refer to Sungur [16]. □

Next, we discuss the concept of a truncated invariant copula based on Theorem 1. For the sake of simplicity, we denote C_{ij}, $i \neq j \in \{1, 2, 3\}$, and C to represent two- and three-dimensional copulas, respectively.

Definition 1. If a three-dimensional copula can be expressed as

$$\begin{aligned} C(u^1, u^2, u^3) &= C_{12}\left(\frac{C_{13}(u^1, u^3)}{u^3}, \frac{C_{23}(u^2, u^3)}{u^3}\right)u^3 \\ &= C_{23}\left(\frac{C_{12}(u^1, u^2)}{u^1}, \frac{C_{13}(u^1, u^3)}{u^1}\right)u^1 \\ &= C_{13}\left(\frac{C_{12}(u^1, u^2)}{u^2}, \frac{C_{23}(u^2, u^3)}{u^2}\right)u^2, \end{aligned}$$

then it is called a truncated invariant copula. If only one of the above equations is satisfied, then it is called a partially truncated invariant copula with respect to the truncation variable.