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Fredholm operators and nonuniform exponential dichotomies☆

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ABSTRACT

We show that the existence of a nonuniform exponential dichotomy for a one-sided sequence $(A_m)_{m\geq 0}$ of invertible $d \times d$ matrices is equivalent to the Fredholm property of a certain linear operator between spaces of bounded sequences. Moreover, for a two-sided sequence $(A_m)_{m\in\mathbb{Z}}$ we show that the existence of a nonuniform exponential dichotomy implies that a related operator S is Fredholm and that if it is Fredholm, then the sequence admits nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$. We also give conditions on S so that the sequence admits a nonuniform exponential dichotomy on $\mathbb{Z}$. Finally, we use the former characterizations to establish the robustness of the notion of a nonuniform exponential dichotomy.

1. Introduction

1.1. From uniform to nonuniform exponential behavior

Our main aim is to discuss the relation between the existence of a *nonuniform* exponential dichotomy for a sequence of invertible $d \times d$ matrices (see Section 2 for the definition) and the Fredholm property of a certain linear operator. Related results were first obtained by Palmer [14,15] for ordinary differential equations and *uniform* exponential dichotomies. Further results were obtained by Lin [11] for functional differential equations, by Blázquez [6], Rodrigues and Silveira [20], Zeng [23] and Zhang [24] for parabolic evolution equations, and by Chow and Leiva [7], Sacker and Sell [21] and Rodrigues and Ruas-Filho [19] for abstract evolution equations. We emphasize that all these works consider only uniform exponential dichotomies.

In comparison to the classical notion of a uniform exponential dichotomy, the notion of a nonuniform exponential dichotomy corresponds to a much weaker requirement. For example, in the context of ergodic theory almost all linear variational equations with nonzero Lyapunov exponents of a measure-preserving flow (such as any Hamiltonian flow restricted to a compact hypersurface) admit a nonuniform exponential dichotomy (see for example [4]). On the other hand, the extra exponentials in the notion of a nonuniform exponential dichotomy (see (5) and (6)) complicate a corresponding study.

In order to circumvent this difficulty we shall use Lyapunov norms (see Proposition 4). These are norms adapted to each particular dynamics with respect to which a nonuniform exponential dichotomy becomes uniform (the crucial properties of the Lyapunov norms are those in (3), (4) and (7)). The use of Lyapunov norms in the study of

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nonuniform hyperbolicity goes back to seminal work of Pesin [16] (see also [4,5]).

Incidentally, an alternative characterization of a nonuniform exponential behavior that does not involve constructing Lyapunov norms a priori was developed in [10] in the context of ergodic theory (when the nonuniform part of the exponential dichotomy can be made arbitrarily small). The characterization is expressed in terms of the invertibility of certain linear operators on Fréchet spaces.

1.2. Brief formulation of our results

As noted above, our main aim is to discuss the relation between the existence of a nonuniform exponential dichotomy for a sequence of matrices and the Fredholm property of a certain linear operator.

More precisely, we consider separately the cases of one-sided and two-sided sequences. Let

$$l^\infty = \left\{ \mathbf{x} = (x_m)_{m\geq 0} \subset \mathbb{R}^d : \sup_{m\geq 0} \|x_m\|_m < +\infty \right\}$$

for some norms $\|\cdot\|_m$ on $\mathbb{R}^d$, for $m \geq 0$, and denote by l_0^∞ the set of all $\mathbf{x} \in l^\infty$ with $x_0 = 0$. Given a sequence $(A_m)_{m\geq 0}$ of invertible $d \times d$ matrices, we define a linear operator $T : \mathcal{D}(T) \to l_0^\infty$ by

$$(T\mathbf{x})_0 = 0 \quad \text{and} \quad (T\mathbf{x})_{m+1} = x_{m+1} - A_m x_m, \quad m \geq 0,$$

in the set $\mathcal{D}(T)$ of all $\mathbf{x} \in l^\infty$ such that $T\mathbf{x} \in l_0^\infty$.

In particular, we establish the following result (see Theorems 6 and 7).

Theorem 1. *The sequence $(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ if and only if T is a Fredholm operator for some norms $\|\cdot\|_m$ satisfying*

$$\|x\| \leq \|x\|_m \leq Ce^{\varepsilon m}\|x\|, \quad m \geq 0, \ x \in \mathbb{R}^d,$$

for some constants $C > 0$ and $\varepsilon \geq 0$.

In order the formulate a corresponding result for two-sided sequences, let

$$l_{\mathbb{Z}}^\infty = \left\{ \mathbf{x} = (x_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^d : \sup_{m\in\mathbb{Z}} \|x_m\|_m < +\infty \right\},$$

for some norms $\|\cdot\|_m$ on $\mathbb{R}^d$, for $m \in \mathbb{Z}$. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d \times d$ matrices, we define a linear operator $S : \mathcal{D}(S) \to l^\infty$ by

$$(S\mathbf{x})_m = x_m - A_{m-1}x_{m-1}, \quad m \in \mathbb{Z},$$

in the set $\mathcal{D}(S)$ of all $\mathbf{x} \in l^\infty$ such that $S\mathbf{x} \in l^\infty$.

We also establish the following version of Theorem 1 for two-sided sequences of matrices (see Theorems 10 and 11).

Theorem 2. *The sequence $(A_m)_{m\in\mathbb{Z}}$ admits nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ if and only if S is a Fredholm operator for some norms $\|\cdot\|_m$ satisfying*

$$\|x\| \leq \|x\|_m \leq Ce^{\varepsilon|m|}\|x\|, \quad m \in \mathbb{Z}, \ x \in \mathbb{R}^d,$$

for some constants $C > 0$ and $\varepsilon \geq 0$.

For two-sided sequences of matrices, we also show that if S is a Fredholm operator and

$$R = S|_E : E \to c_0$$

is injective, where $E = S^{-1}c_0$ and

$$c_0 = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^\infty : \lim_{|n|\to+\infty} \|x_n\|_n = 0 \right\}, \tag{1}$$

then the sequence of matrices admits a nonuniform exponential dichotomy on the whole $\mathbb{Z}$ (see Theorem 11).

As an immediate consequence of Theorems 1 and 2, by considering a constant sequence of norms $\|\cdot\|_m = \|\cdot\|$ on $\mathbb{R}^d$ we obtain the following discrete-time version of results of Palmer [14,15]. However, in contrast to what happens in his work (for continuous time), we do not require any bounded growth condition for the matrices A_m.

Theorem 3. *The following properties hold:*

1. *$(A_m)_{m\geq 0}$ admits a uniform exponential dichotomy on $\mathbb{Z}_0^+$ if and only if T is a Fredholm operator taking $\|\cdot\|_m = \|\cdot\|$ for $m \geq 0$;*
2. *$(A_m)_{m\in\mathbb{Z}}$ admits uniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ if and only if S is a Fredholm operator taking $\|\cdot\|_m = \|\cdot\|$ for $m \in \mathbb{Z}$.*

We are not able to provide a reference for Theorem 3, but it certainly should be considered a folklore result in the area (although perhaps adding a bounded growth condition).

1.3. Application to robustness

In addition, we use the characterization of the existence of a nonuniform exponential dichotomy, both for one-sided and two-sided sequences of matrices, to establish the robustness of the notion (see Theorems 8 and 12). We note that the study of robustness has a long history. In particular, it was discussed by Massera and Schäffer [12], Coppel [8], and in the case of Banach spaces by Dalec′kiĭ and Kreĭn [9]. For more recent works we refer to [13,17,18] and the references therein. We refer the reader to [5] and the references therein for the study of robustness of a nonuniform exponential behavior.

As noted above, in order to circumvent the difficulty caused by the extra exponentials in the notion of a nonuniform exponential dichotomy we use Lyapunov norms. Thus, one might think that we would always need to know the Lyapunov norms a priori in order to be able to use our results. Of course this depends on the particular context, although our proof of the robustness of a nonuniform exponential dichotomy (both for one-sided and two-sided sequences of matrices) shows that sometimes there is no need whatsoever to know explicitly the Lyapunov norms in order to apply our results.

2. Preliminaries

Given $I \in \{\mathbb{Z}_0^+, \mathbb{Z}_0^-, \mathbb{Z}\}$, where

$$\mathbb{Z}_0^+ = \{m \in \mathbb{Z} : m \geq 0\} \quad \text{and} \quad \mathbb{Z}_0^- = \{m \in \mathbb{Z} : m \leq 0\},$$

we consider a sequence $(A_m)_{m\in I}$ of invertible $d \times d$ matrices and norms $\|\cdot\|_m$ on $\mathbb{R}^d$ for $m \in I$. For each $m, n \in I$