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Emergence of a multilayer structure in adaptive networks of phase oscillators

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ABSTRACT

We report on self-organization of adaptive networks, where topology and dynamics evolve in accordance to a competition between homophilic and homeostatic mechanisms, and where links are associated to a vector of weights. Under an appropriate balance between the intra- and inter- layer coupling strengths, we show that a multilayer structure emerges due to the adaptive evolution, resulting in different link weights at each layer, i.e. different components of the weights' vector. In parallel, synchronized clusters at each layer are formed, which may overlap or not, depending on the values of the coupling strengths. Only when intra- and inter- layer coupling strengths are high enough, all layers reach identical final topologies, collapsing the system into, in fact, a monolayer network. The relationships between such steady state topologies and a set of dynamical network's properties are discussed.

Coupled biological and chemical systems, social groups and interacting animal species, the Internet and the World Wide Web, the brain and the stock markets are just a few examples of systems composed of a huge number of highly interconnected dynamical components. The modern approach to capture the global properties of such systems is to model them as graphs [1–4], where nodes represent the basic units, and links stand for the interactions between them, forming a specific connectivity pattern which defines the so-called network's topology. Despite their intrinsic differences, a set of surprising common properties, such as a power law scaling in the network connectivity and the coexistence of modules observed at the mesoscopic scale, has been revealed in real-world network (RWN) [5]. The spontaneous emergence of these topological features has been recently explained as a consequence of a self-organization process involving structure-dynamics adaptation of two fundamental mechanisms [6,7]. The first one corresponds to the trend of reinforcing those interactions with other correlated units in the network, which is a well established process known as *homophily* in the case of social systems [8] and Hebbian learning in the field of neuroscience [9]. The second process results instead from the limitation of the associative capacity, which preserves

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the value of the inputs/outputs received by each unit. This mechanism is known as *homeostasis* [10] in neuroscience, while in social systems it is related to the so-called Dunbar's number [11], which explains the existence of a maximum number of interactions for an individual.

Up until recently, attention was almost exclusively concentrated on networked systems where all components were treated on an equivalent footing, while neglecting all the extra information about the temporal- or context-related properties of RWNs' interactions. Only in the last years, and taking advantage of an enhanced resolution in real data sets, the interest switched to properly frame the *multilayer* character of RWNs, by considering them as networks made of diverse relationships (layers) between their constituents [12,13]. The analysis of multilayer networks started with a reformulation of classical topological parameters, such as the shortest path length, clustering coefficient, centrality or robustness of the nodes [14–17]. From the dynamical perspective, the multilayer formulation has been applied both to networks whose layers coexist or alternate in time [13]. In both cases, the multilayer formulation allows to identify synchronization regions that arise as a consequence of the interplay between the layers' topologies [18–20], as well as to define new types of synchronization based on the coordination between layers [21].

In this paper, we focus on how the competition between homophily and homeostasis can actually lead to self-organization of ensembles of oscillators into a multilayer network structure. To this purpose, we will consider a generic adaptive network of phase oscillators, and report the way a multilayer structure of interactions emerges and is maintained when the weights of the network's connections evolve according to the dynamical properties of the nodes and, conversely, how the evolution of the network topology influences the dynamics of the nodes and their ability to synchronize. Particularly, we make use of an extension of the classical Kuramoto model [22] as a paradigmatic phase oscillator able to describe the dynamics of a series of physical, biological, technological and social systems [23,24]. This way we are able to investigate the interplay between the generic dynamics of phase oscillators and the evolution of the structure where the dynamical units are constrained to interact.

Our starting point is, then, an ensemble of N oscillators whose dynamics evolves in time. Each oscillator i $(i = 1, \dots, N)$ has a natural frequency ω_i, and, in order to encompass the most general case, it is described by a *phasor* $\vec{\phi}_i$, i.e. a vector of M components ϕ_i^l $(l = 1, \dots, M)$ which actually stand for its instantaneous, time dependent, phases in each of the M layers of the multilayer network on which the oscillator interacts with the rest of the ensemble. For the sake of simplicity, we assume a Kuramoto-like evolution for the phase $\phi_i^l(t)$ on each layer $l = 1, \dots, M$. Our choice is motivated by the fact that the interaction of Kuramoto oscillators is a paradigm of synchronization in nonlinear science [22], and actually represents (though in its simplicity) a rather elegant way to encompass synchronous phenomena occurring in many biological (such as circadian clocks), technological (electrical generators), and social systems (opinion formation). Furthermore, for each oscillator, we model layer-layer interactions by an additional coupling term accounting for the rigidity of the phasor, i.e. implying all-to-all interactions between the different components of the vector $\vec{\phi}_i$. The resulting evolution of the phasors is given by

$$\dot{\phi}_i^l(t) = \omega_i + \sigma_1 \sum_{j \neq i} w_{ij}^l(t) \sin(\phi_j^l - \phi_i^l) + \sigma_2 \sum_{j \neq l} \sin(\phi_i^j - \phi_i^l). \quad (1)$$

Here, $\{\omega_i\}$ is a set of randomly assigned natural frequencies distributed uniformly in $[-\pi, \pi]$ (note that the natural frequency ω_i of ith oscillator is the same for all M layers of the network), and σ_1 and σ_2 are the intra- and inter-layer coupling strengths, respectively.

This way, the exchange of information of the dynamical state of each layer relies on the interaction of the phases within the same oscillator i, which is controlled by the inter-layer coupling σ_2.

On the other hand, $w_{ij}^l(t)$ is the weight of the connection between elements i and j on layer l and it is allowed to evolve in time, e.g. layers are allowed to reorganize internally. On each layer l, for each oscillator i and at each time t, the set of connection weights $\{w_{ij}^l\}$ satisfies the condition

$$\sum_{j \neq i}^{N} w_{ij}^l = 1. \quad (2)$$

In other words, we consider the case for which, in Eq. (2), the input strength received by each unit i within each layer is constant, as in homeostatic processes [21].

In parallel with the node dynamics given by Eq. (1), the weights of the links are also evolving following differential equations that reflect a competition between homophily and homeostasis [6,7]. The adaptive evolution of the weights w_{ij}^l is governed by

$$\dot{w}_{ij}^l(t) = p_{ij}^l(t) - \left(\sum_{k \neq i} p_{ik}^l(t) \right) w_{ij}^l(t), \quad (3)$$

where the time dependent quantity $p_{ij}^l(t)$ is defined as

$$p_{ij}^l(t) = \frac{1}{T} \left| \int_{t-T}^{t} e^{i(\phi_i^l(t') - \phi_j^l(t'))} dt' \right|. \quad (4)$$

Notice that p_{ij}^l denotes, at time t, the average phase correlation (within layer l) between oscillators i and j over a characteristic memory time T. It follows from Eq. (3) that the normalization condition given by Eq. (2) holds at all times, i.e., the sum of the weights of all incoming connections at each node within each layer is conserved.

The particular case of a monoplex $(M = 1)$ was extensively studied in Refs. [6,7], both numerically and analytically, and it was shown that a large region exists in the parameter space (σ_1, T) where, starting from random initial conditions for the weights w_{ij}^1 and phases ϕ_i^1, the network asymptotically reaches a state organized in synchronous clusters. Within this regime, the global phase coherence is rather small while the local coherence (i.e. the level of phase synchronization of each oscillator within its neighborhood) is very high, showing, at the same time, a scale-free distribution of the connection weights w_{ij}^1 as $t \to \infty$.