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Kato's chaos in duopoly games

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ABSTRACT

Let $E, F \subset \mathbb{R}$ be two given closed intervals, and let $\tau\colon E \to F$ and $\theta\colon F \to E$ be continuous maps. In this paper, we consider Koto's chaos, sensitivity and accessibility of a given system $\Psi(u, v) = (\theta(v), \tau(u))$ on a given product space $E \times F$ where $u \in E$ and $v \in F$. In particular, it is proved that for any Cournot map $\Psi(u, v) = (\theta(v), \tau(u))$ on the product space $E \times F$, the following hold:

(1) If Ψ satisfies Kato's definition of chaos then at least one of $\Psi^2|_{Q_1}$ and $\Psi^2|_{Q_2}$ does, where $Q_1 = \{(\theta(v), v) : v \in F\}$ and $Q_2 = \{(u, \tau(u)) : u \in E\}$.
(2) Suppose that $\Psi^2|_{Q_1}$ and $\Psi^2|_{Q_2}$ satisfy Kato's definition of chaos, and that the maps θ and τ satisfy that for any $\varepsilon > 0$, if

$$| (\tau \circ \theta)^n(v_1) - (\tau \circ \theta)^n(v_2) | < \varepsilon$$

and

$$| (\theta \circ \tau)^m(u_1) - (\theta \circ \tau)^m(u_2) | < \varepsilon$$

for some integers n, $m > 0$, then there is an integer $l(n, m, \varepsilon) > 0$ with

$$| (\tau \circ \theta)^{l(n,m,\varepsilon)}(v_1) - (\tau \circ \theta)^{l(n,m,\varepsilon)}(v_2) | < \varepsilon$$

and

$$| (\theta \circ \tau)^{l(n,m,\varepsilon)}(u_1) - (\theta \circ \tau)^{l(n,m,\varepsilon)}(u_2) | < \varepsilon.$$

Then Ψ satisfies Kato's definition of chaos.

1. Introduction and preliminaries

Let $E, F \subset \mathbb{R}$ be closed intervals and $\theta\colon F \to E$ and $\tau\colon E \to F$ be continuous maps, and let $\Psi\colon E \times F \to E \times F$ be defined as $\Psi(u, v) = (\theta(v), \tau(u))$ for any $(u, v) \in E \times F$. Such maps have been studied to present a mathematical analysis of Cournot duopoly (see [1]). Probably the first definition of chaos in a mathematically rigorous way is given by Li and Yorke [2]. Since then, a lot of different definitions of chaos have been given. Each of them is used to reflect some kind of unpredictability of a given system in the evolution of this system. Akin and Kolyada defined Li–Yorke sensitivity for the first time [3]. Moreover, they defined spatio-temporal chaos. Schweizer and Smítal defined distributional chaos [4]. It is noted that distributional chaos is equivalent to positive topological entropy and some other definitions of chaos for some spaces (see [4,5]), and that this equivalence relationship does not hold for higher dimensional spaces [6] and zero-dimensional spaces [7]. In [8], Wang et al. defined distributional chaos with respect to a sequence and established that this kind of chaos is

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equivalent to Li–Yorke's chaos for continuous self-maps on a closed interval. Over the past few decades, people always paid very close attention to the dynamic behavior of Cournot maps (see [1,9–13]). From [1,12] we know that there exist Markov perfect equilibria processes. Concretely speaking, two fixed players move alternatively such that each of them chooses the best reply to the previous action of another player. Let $Q_1 = \{(\theta(v), v) : v \in F\}$, $Q_2 = \{(u, \tau(u)) : u \in E\}$ and $Q_{12} = Q_1 \cup Q_2$. Clearly, $\Psi(Q_{12}) \subset Q_{12}$. Q_{12} is called a MPE-set for Ψ (see [9]). Moreover, in [9] the authors explored several definitions of chaos for Cournot maps, and showed that for any definition they considered in [9] it does not satisfy that Ψ is chaotic if and only if so is $\Psi|_{Q_{12}}$. Note that some chaotic properties of Cournot maps have been studied (see [1,12–17]). Recently, Lu and Zhu further investigated a few chaotic properties of Cournot maps. In particular, it was proved that for $\Psi|_{Q_{12}}$, $\Psi^2|_{Q_1}$ and $\Psi^2|_{Q_2}$, they enjoy some same chaotic properties. Motivated by Cánovas and Marin [9], and Lu and Zhu [13], we will continue to explore the chaotic properties of the above Cournot maps. In particular, we prove that for any Cournot map $\Psi(u, v) = (\theta(v), \tau(u))$ on the product space $E \times F$, the following hold:

(1) If Ψ satisfies Kato's definition of chaos then at least one of $\Psi^2|_{Q_1}$ and $\Psi^2|_{Q_2}$ does, where $Q_1 = \{(\theta(v), v) : v \in F\}$ and $Q_2 = \{(u, \tau(u)) : u \in E\}$.

(2) Suppose that $\Psi^2|_{Q_1}$ and $\Psi^2|_{Q_2}$ satisfy Kato's definition of chaos, and that the maps θ and τ satisfy that for any $\varepsilon > 0$, if

$$| (\tau \circ \theta)^n (v_1) - (\tau \circ \theta)^n (v_2) | < \varepsilon$$

and

$$| (\theta \circ \tau)^m (u_1) - (\theta \circ \tau)^m (u_2) | < \varepsilon$$

for some integers n, $m > 0$, then there is an integer $l(n, m, \varepsilon) > 0$ with

$$| (\tau \circ \theta)^{l(n,m,\varepsilon)} (v_1) - (\tau \circ \theta)^{l(n,m,\varepsilon)} (v_2) | < \varepsilon$$

and

$$| (\theta \circ \tau)^{l(n,m,\varepsilon)} (u_1) - (\theta \circ \tau)^{l(n,m,\varepsilon)} (u_2) | < \varepsilon.$$

Then Ψ satisfies Kato's definition of chaos.

Let (E, e) be a metric space.

A dynamic system (E, θ) or a map $\theta: E \to E$ is transitive if for any nonempty open subsets $S, T \subset E$, $\theta^m(S) \cap T \neq \emptyset$ for some integer $m > 0$.

A dynamic system (E, θ) or a map $\theta: E \to E$ is topologically mixing if for any nonempty open subsets $S, T \subset E$, one can find an integer $n > 0$ such that $\theta^m(S) \cap T \neq \emptyset$ for any integer $m \geq n$.

A dynamic system (E, θ) or a map $\theta: E \to E$ is sensitive if there is a $\lambda > 0$ such that for any given $\alpha > 0$ and any given $u \in E$, there is a point $v \in S$ with $e(u, v) < \alpha$ and $e(\theta^m(u), \theta^m(v)) > \lambda$ for some integer $m > 0$, where λ is called a sensitivity constant of θ.

A dynamic system (E, θ) or a map $\theta: E \to E$ is accessible if for any $\lambda > 0$ and any two nonempty open subsets S, $T \subset E$, there are two points $u \in S$ and $v \in T$ with $e(\theta^m(u), \theta^m(v)) < \lambda$ for some integer $m > 0$.

A dynamic system (E, θ) or a map $\theta: E \to E$ is chaotic in the sense of Ruelle and Takens [18] if it is transitive and sensitive. A dynamic system (E, θ) or a map $\theta: E \to E$ satisfies Kato's definition of chaos if it is sensitive and accessible. Note that a topologically mixing dynamic system (E, θ) or a topologically mixing map $\theta: E \to E$ satisfies Kato's definition of chaos [19]. We will need the following lemma which is from [13].

Lemma 1.1. *(see [13], Theorem 3.1]) Let $J_1 \subset \mathbb{R}$ and $J_2 \subset \mathbb{R}$ be two compact intervals and $g_j: J_j \to J_j$ be a continuous map for every $j \in \{1, 2\}$. Then $g_1 \times g_2$ is $\mathscr{P}$-chaotic if and only if either g_1 or g_2 is $\mathscr{P}$-chaotic, where $\mathscr{P}$ denotes one of the following five properties: Li–Yorke chaos, distributional chaos in a sequence, Li–Yorke sensitivity, sensitivity and distributional chaos.*

2. Main results

The following two lemmas are needed.

Lemma 2.1. *Let (E, e) be a compact metric space and $\theta: E \to E$ be a continuous map. Then θ is sensitive if and only if so is θ^2.*

Proof. Clearly, if θ^2 is sensitive then so is θ. If f is sensitive then by the definition there is a sensitivity constant $\lambda' > 0$ of θ. As θ is uniformly continuous, there is $\lambda \in (0, \lambda')$ such that $e(u, v) \leq \lambda$ with $u, v \in E$ implies $e(\theta(u), \theta(v)) \leq \lambda'$. By the definition, for any given $\alpha > 0$ and any given $u \in E$, there is a point $v \in E$ with $e(u, v) < \alpha$ and $e(\theta^m(u), \theta^m(v)) > \varepsilon'$ for some integer $m > 0$. Then, it is easily seen that if $m = 2k + 1$ then $e(\theta^{2k}(u), \theta^{2k}(v)) > \lambda' > \lambda$, and that if $m = 2k$ then $e(\theta^{2k}(u), \theta^{2k}(v)) > \lambda$. By the definition, θ^2 is sensitive. □

Lemma 2.2. *Let (E, e) be a compact metric space and $\theta: E \to E$ be a continuous map. Then θ is accessible if and only if so is θ^2.*

Proof. Obviously, if θ^2 is accessible then so is θ. Now, we suppose that θ is accessible. As θ is uniformly continuous, for any $\lambda > 0$ there is $\lambda' \in (0, \lambda)$ such that $e(u, v) < \lambda'$ with $u, v \in E$ implies $e(\theta(u), \theta(v)) < \lambda$. By the definition, for any two nonempty open subsets S, $T \subset E$ and the above $\lambda' > 0$, there are two points $u \in S$ and $v \in T$ and an integer $m > 0$ with $e(\theta^m(u), \theta^m(v)) < \lambda'$. Then, it is easily seen that if $m = 2k - 1$ then $e(\theta^{2k}(u), \theta^{2k}(v)) < \varepsilon$, and that if $m = 2k$ then $e(\theta^{2k}(u), \theta^{2k}(v)) < \lambda' < \lambda$. By the definition, θ^2 is accessible. □

Theorem 2.1. *Let (E, e) be a compact metric space and $\theta: E \to E$ be a continuous map. Then θ satisfies Kato's definition of chaos if and only if so does θ^2.*

Proof. By Lemmas 2.1, 2.2, and Theorem 2.1 is true. Thus, the proof is ended. □

In [13], the authors proved that for a given Cournot map Ψ, Ψ is Li–Yorke sensitive if and only if so is one of the maps $\Psi^2|_{Q_1}$ and $\Psi^2|_{Q_2}$. Inspired by this result, we have the following three basic results.