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Explosions in Lorenz maps

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ABSTRACT

We introduce a map to describe the systematics of orbit creation and annihilation in Lorenz-like dynamical systems. This map, $y' = b - \sqrt{|y|}$, has a singular maximum and is useful for describing flows that undergo a tear-and-squeeze route to chaos. We call this map the Lorenz map. We find: much of the dynamics is determined by the bifurcations of the period-one and period-two orbits; orbits are created in explosions (singular saddle-node bifurcations) based on two symbols s_0, s_1, and later removed in inverse processes that are implosions. The order in which direct and inverse explosions occur generally follows the inverse order shown by the logistic map. In the entire parameter range only one regular saddle-node bifurcation and one period-doubling bifurcation occurs.

1. Introduction

The logistic map adequately describes dissipative flows that generate chaotic behavior by a stretch-and-fold mechanism. One such flow is defined by the Rössler equations [1]. For such flows the return map on a Poincaré section typically exhibits a parabolic shape in the limit of large dissipation. This shape comes about for the following reason. Initial conditions near an unstable focus are first accelerated away from the fixed point, but then must decelerate before being returned to the neighborhood of the unstable fixed point. The deceleration is responsible for the parabolic shape of the return map.

Other flows, such as that associated with the Lorenz equations [2], generate chaos by a different mechanism. This is the tear-and-squeeze mechanism [3,4]. Return maps for such flows do not exhibit a quadratic extremum: rather their extrema are singular. One such return map appears in [2], Fig. 4. This shape comes about because points in the neighborhood of an unstable focus are accelerated away from that focus even as they fall into a domain largely influenced by a different focus. This continued acceleration is responsible for the non differentiable nature of the return map. Flows that exhibit this type of return map occur in fluids [5,6] and in lasers [7]. In particular, natural systems whose underlying dynamics has a twofold symmetry and exhibits chaotic behavior typically exhibit this phenomenon.

A great deal of information about orbit creation and annihilation in the Rössler and similar systems can be determined by detailed study of the logistic map.

In order to understand the systematics of orbit creation and annihilation in flows exhibiting a tear-and-squeeze mechanism, it is useful to study first return maps with singular maxima. A family of such maps can be introduced that have a structure similar to that of one common version of the logistic map:

$$\begin{aligned} &\text{logistic map } x' = f(x;a) = a - (|x|)^2 \\ &\text{Lorenz map } y' = g(y;b) = b - (|y|)^{1/2} \end{aligned} \qquad (1)$$

The Lorenz map has two leaves, or branches, separated by a singular point at $y = 0$. We label the branch on the left with positive slope 0 and that on the right with negative slope 1. The form of the separating maximum (x^2 or $\sqrt{|y|}$) has no impact on either the representation of a trajectory by symbols [8] or the kneading invariant [9] of a trajectory or periodic orbit. These invariants are computed for the Lorenz map as for the logistic map.

The logistic map is concave down. That means that a straight line connecting any two points on this map lies

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entirely below the map. By contrast, each of the two monotonic branches of the Lorenz map is concave up. This topological difference introduces some pronounced differences between the two maps. The similarities resulting from the monotonicity of the branches as well as the applicability of the kneading theory, and the differences resulting from topological effects (concavity up or down) will be described in this work.

The organization of this paper is as follows. The bifurcation diagram for the Lorenz map is described in Section 2. The bifurcation diagram is largely constrained by bifurcations involving the period-one and period-two orbits. These are described in Sections 3 and 4. The boundaries of the attractor and the basin of attraction are described in Section 5. The first explosion occurs at $b = 0$ and creates two unstable period-one orbits. It simultaneously creates all trajectories that can be constructed by arbitrary combinations of the two symbols $s_0 = 0$ and $s_1 = 1$. This explosion, which serves as the prototype for all other explosions and implosions exhibited by this map, is described in Section 6. The saddle-node bifurcations that occur in the logistic map are replaced by "singular saddle-node bifurcations" in the Lorenz map. Unstable periodic orbits are created in the explosion that occurs at $b = 0$ and removed in a series of implosions as b increases, with the last implosion occurring at $b = 1$. An implosion as b increases through a critical value b_{cr} appears as an explosion as b decreases through b_{cr}. In Section 7 we describe why caustics are not visible in the bifurcation diagram of the Lorenz map while they dominate the bifurcation diagram of the logistic map. The zero crossings of the anticaustics of the Lorenz map provide a useful tool for locating explosions involving primary orbits of various periods and also for determining the order in which the explosions take place. This order is described in Section 8. The analog of the period three window of the logistic map, involving an inverse singular saddle-node bifurcation, is described in Section 9. Other implosions are described in Section 10. The final implosion, involving two period-two orbits, is described in Section 11. On the path to the final explosion there is a series of implosions and explosions among orbits of high even period ($p \geqslant 14$). These occur in matched pairs and serve to leave the spectrum of trajectories present at the beginning of this interval (b_b) identical to the spectrum at the end (b_c). These processes are discussed in Section 12. The Lorenz map is one in a larger class ($y' = b - |y|^k, 0 < k < 1$) of mappings with singular critical points. The properties of these maps are determined by the same set of critical values of the control parameter as is the Lorenz map. These critical parameters are described in Section 13. The systematic decrease in the topological entropy as b increases for this entire class of maps is displayed in Section 14. We summarize our results in Section 15.

2. Bifurcation Diagram

A great deal of information about the properties of the logistic map $x' = a - x^2$ can be determined from its bifurcation diagram. This shows that a stable period-one orbit is created at $x = -1/2$ when $a = -1/4$. This orbit remains stable for a increasing until a period-doubling bifurcation occurs at $(x, a) = (1/2, 3/4)$. The initial period-doubling bifurcation is followed by a series of period-doubling bifurcations that accumulate at $a_\infty = 1.401155\ldots$. Beyond the accumulation point there is a series of noisy period-halving bifurcations [10] that end at $a_b = 1.543689\ldots$. For $a > a_\infty$ there is a mixture of periodic windows and chaotic behavior that exists until a crisis occurs at $a = 2$. In the range $a_\infty < a < a_b$ all periodic orbits have even period. For $a > 2$ almost all initial conditions escape to a stable fixed point at $x = -$"∞". Only unstable periodic orbits are present in the neighborhood formerly occupied by the chaotic attractor.

By contrast, the bifurcation diagram of the Lorenz map $y' = b - \sqrt{|y|}$ conveys very little information. Fig. 1 shows that there is a chaotic attractor in the range $\frac{1}{4} < b < \frac{3}{4}$ and outside this range there is a stable period-one orbit. These two bounds are the b values at which the only two normal bifurcations exhibited by this map occur: an inverse saddle node bifurcation that occurs at $b_{snb} = \frac{1}{4}$ that destroys a stable period-one orbit; and a regular pitchfork or period-doubling bifurcation that occurs at $b_{pdb} = \frac{3}{4}$ and transforms an unstable period-one orbit into a stable period-one orbit. There are infinitely many periodic orbits in the attractor. The bifurcation diagram does not show that there are very many (countable infinity of) unstable periodic orbits in the intervals $0 < b < b_{snb}$ and $b_{pdb} < b < 1$. The number of periodic orbits decreases in a systematic way as b increases from b_{snb} to b_{pdb}. However, as all bifurcations involve unstable orbits, none appear in the bifurcation diagram. There is a global repellor at $y = -$"∞" for all values of b.

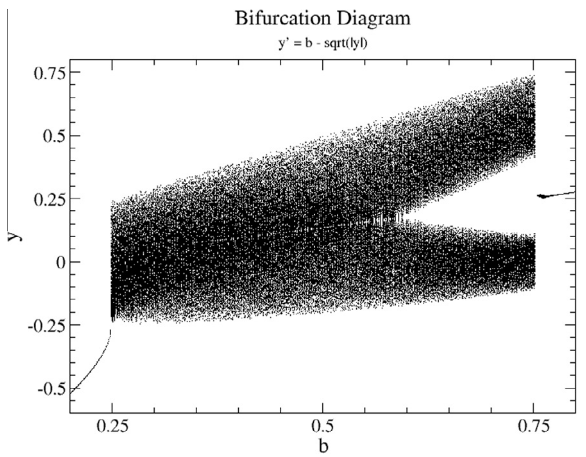

Fig. 1. Bifurcation diagram for the Lorenz map shows globally attracting period one orbits for $b < b_{snb} = \frac{1}{4}$ and $\frac{3}{4} = b_{pdb} < b$. Between these values there is a chaotic attractor exhibiting no stable periodic windows but containing many unstable periodic orbits. The attractor boundaries in this region are $y_+(b) = b$ and $y_-(b) = b - \sqrt{b}$. In the range $b_b \simeq 0.59\ldots < b < b_{pdb}$ all periodic orbits in the attractor have even period except for an unstable period-one orbit. In the ranges $0 < b < b_{snb}$ and $b_{pdb} < b < 1$ there are many unstable periodic orbits.