Chaos, Solitons & Fractals 76 (2015) 182–189

Contents lists available at ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Discovering independent parameters in complex dynamical systems

Claudia Lainscsek [a,b,*], Jonathan Weyhenmeyer [a,c], Terrence J. Sejnowski [a,b], Christophe Letellier [d]

[a] The Salk Institute for Biological Studies, 10010 North Torrey Pines Road, La Jolla, CA 92037, USA
[b] Institute for Neural Computation, University of California at San Diego, La Jolla, CA 92093, USA
[c] Indiana University School of Medicine, Indianapolis, IN 46204, USA
[d] CORIA UMR 6614 – Normandie University, CRNS-Université et INSA de Rouen, Campus Universitaire du Madrillet, F-76800 Saint-Etienne du Rouvray, France

ARTICLE INFO

Article history:
Received 17 February 2015
Accepted 2 April 2015
Available online 28 April 2015

ABSTRACT

The transformation of a nonlinear dynamical system into a standard form by using one of its variables and its successive derivatives can be used to identify the relationships that may exist between the parameters of the original system such as the subset of the parameter space over which the dynamics is left invariant. We show how the size of the attractor or the time scale (the pseudo-period) can be varied without affecting the underlying dynamics. This is demonstrated for the Rössler and the Lorenz systems. We also consider the case when two Rössler systems are unidirectionally coupled and when a Lorenz system is driven by a Rössler system. In both cases, the dynamics of the coupled system is affected.

1. Introduction

Coupling dynamical systems is a very popular approach for understanding how synchronization and control can occur in the natural world. Network of coupled dynamical systems are ubiquitous throughout nature including biological systems (see [1–4] and references therein). It is therefore essential to better understand how coupled nonlinear systems interact, that is, drive each other, and how such interactions affect the collective dynamics. This is particularly relevant for complex networks when selecting for the terms of greatest influence on the collective dynamics and, consequently, which parameters most efficiently affect the network dynamics. This is dual to the observability problem of complex networks [5,6].

In the context of global modeling dynamical systems can be rewritten in a "standard form" [7] (now also called a jerk form [8–12]), that is, by using one of the variables of the original system and its Lie derivatives. When the Ansatz library is used for rewriting a dynamical system in a standard form [13,14], it is possible to identify sets of parameter values or, equivalently, domains in the parameter space, for which the dynamics is invariant; the dynamics is thus left unchanged under a variation of well chosen parameter values as shown for the Rössler system when variable y and its successive Lie derivatives are used [15] or for the Lorenz-like systems when variable x is used [16].

In the present paper, our aim is to determine how many independent parameters can be identified in the Rössler and the Lorenz systems and how they can be affected by the variable retained for the analysis. Moreover, in both cases, we determine how it is possible to vary the pseudo-period of the system by modifying a subset of the system parameters, a consequence previously not considered [13].

We will then consider the case where two Rössler systems are coupled and the case where a Rössler system is used for driving a Lorenz system without affecting the

* Corresponding author at: The Salk Institute for Biological Studies, 10010 North Torrey Pines Road, La Jolla, CA 92037, USA.

http://dx.doi.org/10.1016/j.chaos.2015.04.001
0960-0779/

standard function when the driven system is still considered in a three-dimensional space. The number of independent parameters of the resulting six-dimensional system is investigated and it is shown how the algebraic structure of the two systems can no longer be considered as independent.

We show here the sets of connected parameters in the Lorenz and the Rössler systems, respectively. We also investigate how connections are modified in the case of two systems unidirectionally coupled. Only parameters that are connected in the individual uncoupled systems can be further connected in the coupled system. Furthermore, parameters that are not connected within the individual systems remain independent in the coupled system.

The paper is organized as follows. Section 2 starts with some background on the interplay between the original system and its standard form. The numbers of independent parameters in the Rössler and in the Lorenz systems are then extensively investigated. Section 3 is devoted to the determination of the parameter to adjust for varying the pseudo-period of these two systems. The case of two coupled Rössler systems is discussed in Section 4 and the case of a Lorenz system driven by a Rössler system is discussed in Section 5. Section 6 gives some conclusions.

2. Lorenz and Rössler systems

2.1. General background

Consider a three-dimensional dynamical system of ordinary differential equations (ODEs) in the form

$$\begin{aligned} \dot{x} &= f_x(x,y,z) \\ \dot{y} &= f_y(x,y,z) \\ \dot{z} &= f_z(x,y,z) \end{aligned} \tag{1}$$

with

$$f_i(x,y,z) = a_{i,0} + a_{i,1}x + a_{i,2}y + a_{i,3}z + a_{i,4}x^2 + a_{i,5}xy + a_{i,6}xz + a_{i,7}y^2 + a_{i,8}yz + a_{i,9}z^2$$

where $i \in \{x,y,z\}$ and monomials are of a polynomial form up to the second-order. It was shown in [7,13,14] that when a small subset of coefficients $a_{i,j}$ is nonzero, system (1) can be transformed into a standard form whose structure is

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= X_3 \\ \dot{X}_3 &= F_s(X_1,X_2,X_3) \end{aligned} \tag{2}$$

where the first variable $X_1 = h(x,y,z) = s$ is some function of the state variables, h being the so-called measurement function. The standard function F_s has a multivariate polynomial form depending on some parameters α_k. Map $\Phi : \mathbb{R}^3(x,y,z) \to \mathbb{R}^3(X_1,X_2,X_3)$ between the original state space and the differential embedding allows one to obtain the standard form (2) from the original system (1). There is an associated map $\phi : \mathbb{R}^p(a_{i,j}) \to \mathbb{R}^q(\alpha_k)$ that expresses parameters α_k's of the standard function F_s in terms of parameters $a_{i,j}$ from the original system.

Lainscsek [15,16] showed that there are algebraically inequivalent dynamical systems in the form (1) which can be transformed to the same standard form (2): it is therefore possible to define a class of equivalent dynamical systems in the sense that from the time series point of view they have an identical variable. Moreover, there is a subspace of the parameter space in which the dynamics is unchanged: therefore, there exists an infinite number of global models corresponding to a single time series.

Parameters involved in relationships defining parameter subspaces in which dynamics remains unchanged are said to be *connected*. This property is important for global modeling techniques [17] since there is an infinite number of dynamical systems of the form (1) that share the same time series; the global model which can be obtained from a given time series is thus not unique. Map ϕ further reveals the connections between parameters from the original form (1) and those from the standard form (2), and the number of non-connected and independent parameters in the original system. Since not all parameters of the original system (1) are independent, it was shown [15] that its pseudo-period can be varied by using specific relationships between some of these parameters.

2.2. Lorenz equations

The Lorenz equations [18]

$$\begin{aligned} \dot{x}_L &= a_{1,1}x_L + a_{1,2}y_L \\ \dot{y}_L &= a_{2,1}x_L + a_{2,2}y_L + a_{2,6}x_L z_L \\ \dot{z}_L &= a_{3,3}z_L + a_{3,5}x_L y_L \end{aligned} \tag{3}$$

can be transformed into a standard form (2) whose function F_s is more or less complicated depending on the "measured" variable [7]. With the choice $X_1 = h(x,y,z) = x_L$ the standard function

$$F_{x_L} = \alpha_1 X_1 + \alpha_2 X_1^3 + \alpha_3 X_2 + \alpha_4 X_1^2 X_2 + \alpha_5 \frac{X_2^2}{X_1} + \alpha_6 Z + \alpha_7 \frac{X_2 X_3}{X_1} \tag{4}$$

is made of $N_d = 7$ terms of which two are rational. The standard function F_{x_L} has therefore two singular terms (corresponding to parameters α_5 and α_7) if $X_1 = 0$. Parameters α_k of this standard function are related to parameters $a_{i,j}$ of the Lorenz system (3) as follows

$$\phi_{x_L} = \left| \begin{array}{lcl} \alpha_1 &=& (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})a_{3,3} \\ \alpha_2 &=& -a_{1,1}a_{2,6}a_{3,5} \\ \alpha_3 &=& -(a_{1,1}+a_{2,2})a_{3,3} \\ \alpha_4 &=& a_{2,6}a_{3,5} \\ \alpha_5 &=& -a_{1,1} - a_{2,2} \\ \alpha_6 &=& a_{1,1} + a_{2,2} + a_{3,3} \\ \alpha_7 &=& 1. \end{array} \right. \tag{5}$$

By introducing the scaling transformation [16]

$$a_{i,j} \to \tilde{a}_{i,j} = \lambda^{p(i,j)} a_{i,j} \tag{6}$$