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On classification of high-order integrable nonlinear partial differential equations

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ABSTRACT

In this study we investigate the existence of soliton solutions for partial differential equations with polynomial nonlinearities. We obtain a classification of high-order integrable systems with two-soliton solution, using the τ-function method. Hierarchies of these equations are represented.

1. Introduction

In nonlinear physics, solitons are determined as solitary waves which interact elastically just like particles do. Similar nonlinear processes occur in different physical phenomena, for example, in ionospheric plasma. Physical properties of plasma may significantly change during active effect on the ionosphere in the result of Earth magnetosphere compression by solar wind. It leads to the fact that wave processes in the ionosphere are described by solitary waves. A special case is the appearance of solitons in the ionospheric plasma. During collisions, such waves interact elastically without attenuation. Many equations, admitting soliton solutions, can be obtained from a hydrodynamic approximation during mathematical modeling of similar systems.

The paper considers the equations which admit soliton solutions. Depending on a soliton solution, it is possible to determine the equation which has a similar solution. The equation class is quite extensive. There are some nonlinear equations which admit N-soliton equations. At present, completely integrable equation hierarchies by Lax, Sawada-Kotera, Kaup Kupershmidt are known [1–8]. The N-soliton solution is a very strong condition which depends on the structure of linear and nonlinear summands of the equation and its coefficients. However, there are equation classes with one- or two-soliton solutions but which do not have N-soliton solution. The condition for two-soliton solution is considered in the paper [9]. Though, the structure of these equations and their hierarchy are not specified there.

Classical example of the equation with the N-soliton solution is the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

Consider some nth order partial differential equation $E(u, u_t, u_x u, u_{xx} u_x, u_x^2, \ldots)$ with polynomial nonlinearities in $(1+1)$ variables. Solution $u(x,t)$ is meromorphic if it has only finite number of poles in its Laurent expansion:

$$u(x,t) = \sum_{k=-m}^{\infty} c_k \phi(x,t)^k \tag{2}$$

where degree m is called a singular order of equation [2,3]. It can be obtained by substitution $u(\xi) = \frac{1}{\xi^m}$, where $\partial_\xi \sim \partial_x$. For the KdV Eq. (1) we have

$$\frac{d^3}{d\xi^3}u(\xi) + 6\left(\frac{d}{d\xi}u(\xi)\right)u(\xi) = (m^2+3m+2)\xi^{-m-3} + 6\xi^{-2m-1} \tag{3}$$

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Equalling poles to 0, we get $m+3=2m+1$, so u has degree $m=2$. Of course this is not the necessary condition for two-soliton solutions (2SS). This condition is used to choose nonlinear terms of the nth order equations.

We denote the nth order partial differential equation of a singular order m as $E_m^n(u)$. Hirota's τ-function [2,10] for $E_m^n(u)$ equation is defined as

$$u(x,t) = K\frac{\partial^m}{\partial x^m}\log\tau(x,t) \tag{4}$$

For example, for the KdV equation one-soliton solution (solitary wave) can be expressed as

$$u(x,t) = K\frac{\partial^2}{\partial x^2}\log(1+\exp(px-qt)) \tag{5}$$

and 2SS

$$\begin{aligned} u(x,t) = K\frac{\partial^2}{\partial x^2}\ln\{&1+\exp(p_1x-q_1t)+\exp(p_2x-q_2t)\\ &+\alpha_{12}\exp((p_1+p_2)x-(q_1+q_2)t)\} \end{aligned} \tag{6}$$

where $\alpha_{12}=\left(\frac{p_1-p_2}{p_1+p_2}\right)^2$, p_i, q_i are constants.

We say that an equation is *partially integrable* if it has a 2SS in Hirota's form. A remarkable fact is that some equations with soliton-type solutions form hierarchies (Lax and Sawada-Kotera equations), i.e. they admit the same type of solitons. For example, famous KdV equation is the first element in the Lax hierarchy.

Our primary goal is to investigate the existence of high order E_2^n equations with two-soliton solutions [1,10,11]. All these equations have terms with ∂_x of singular order $m=2$ and only one term with ∂_t

$$\begin{aligned} E_2^3(u) &= u_t + a_1uu_x + u_{xxx}\\ E_2^5(u) &= u_t + a_1uu_{xxx} + a_2u^2u_x + a_3u_xu_{xx} + u_{xxxxx}\\ E_2^7(u) &= u_t + a_1u_x^3 + a_2uu_xu_{xx} + a_3uu_{xxxxx} + a_4u^2u_{xxx}\\ &\quad + a_5u^3u_x + a_6u_xu_{xxxx} + a_7u_{xx}u_{xxx} + u_{xxxxxxx} \end{aligned}$$

and so on, for arbitrary large n. We want to know if there are other equations with the same singular order of terms, but they differ from the Lax and Sawada-Kotera hierarchies.

2. Soliton solutions for E_2^n

We take a τ-function expansion

$$\tau = 1+\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3+\cdots, \tag{7}$$

where ε is an arbitrary complex parameter. Then we obtain substitution for 2SS solution [1,10,12]:

$$f_1(x,t) = \exp(p_1x-q_1t)+\exp(p_2x-q_2t) \tag{8}$$
$$f_2(x,t) = \alpha_{12}\exp((p_1+p_2)x-(q_1+q_2)t) \tag{9}$$
$$f_i(x,t) = 0,\quad i\geqslant 3 \tag{10}$$

$\alpha_{12}=\frac{R_n(p_1,q_1,p_2,q_2)}{S_n(p_1,q_2,p_2,q_2)}$ is a rational function of complex parameters p_i, q_i. After substituting (8)–(10) into E_2^n we equal the coefficients at powers of ε. The resulting algebraic system depends on a_i, α_{12} coefficients for each order n. Direct computation of the linear part $u_{nx}+u_t$ leads to $q_i=p_i^n$, which is called a dispersion relation [1]. So we are going to obtain explicit form for denominator S_n of α_{12}, when $q_i=p_i^n$, $i=1,2$.

With

$$\frac{\partial^n}{\partial x^n}+u\frac{\partial}{\partial x}+\frac{\partial}{\partial t},\quad n=3,5,7,\ldots \tag{11}$$

where $u(x,t)$ is 2-soliton substitution of (7) by (8)–(10), we equal coefficients at ε. Calculating the coefficient at ε^2 from the expansion above, we get polynoms S_n.

$$\begin{aligned} S_3 &= 3(p_1+p_2)^2\\ S_5 &= 5(p_1+p_2)^2(p_1^2+p_1p_2+p_2^2)\\ S_7 &= 7(p_1+p_2)^2(p_1^2+p_1p_2+p_2^2)^2\\ S_9 &= 3(p_1+p_j)^2\left(3p_1^6+9p_1^5p_2+19p_1^4p_j^2+23p_1^3p_j^3\right.\\ &\quad\left.+19p_1^2p_2^4+9p_1p_2^5+3p_2^6\right)\\ S_{11} &= 11(p_1+p_2)^2({p_1}^2+p_1p_2+p_2^2)({p_j}^6+3p_1p_2^5\\ &\quad+7p_1^2p_2^4+9p_i^3p_2^3+7p_1^4p_2^2+3p_1^5p_2+p_1^6)\\ S_{13} &= 13(p_1+p_2)^2({p_1}^2+p_1p_2+p_2^2)^2\\ &\quad\times\left({p_2}^6+3p_1p_2^5+8p_1^2p_j^4+11p_1^3p_2^3+8p_1^4p_2^2+3p_1^5p_2+p_1^6\right)\\ S_{15} &= (p_1+p_j)^2(15p_1^{12}+15p_2^{12}+90p_2p_1^{11}+90p_2^{11}p_1\\ &\quad+365p_2^{10}p_1^2+1000p_j^9p_1^3+2003p_2^8p_1^4+3002p_j^7p_1^5\\ &\quad+3433p_2^6p_1^6+3002p_2^5p_1^7+2003p_2^4p_1^8+1000p_2^3p_i^9+365p_2^2p_1^{10}) \end{aligned}$$

These polynomials can be derived by horizontal summation of neighbor elements in a binomial triangle, as shown below.

$$\begin{array}{c} \mathbf{1},\mathbf{1}\\ \mathbf{1},3,3,\mathbf{1}\\ \mathbf{1},5,10,10,5,\mathbf{1}\\ \mathbf{1},7,21,35,35,21,7,\mathbf{1}\\ \mathbf{1},9,36,84,126,126,84,36,9,\mathbf{1}\\ \mathbf{1},11,55,165,330,462,462,330,165,55,11,\mathbf{1}\\ \mathbf{1},13,78,286,715,1287,1716,1716,1287,715,286,78,13,\mathbf{1}\\ \Downarrow\\ 3,6,3\\ 5,15,20,15,5\\ 7,28,56,70,56,28,7\\ 9,45,120,210,252,210,120,45,9\\ 11,66,220,495,792,924,792,495,220,66,11\\ 13,91,364,1001,2002,3003,3432,3003,2002,1001,364,91,13 \end{array}$$

For every integer n roots of S_n lie on a curve defined by the following way (see Fig. 1): $Re(z)\in\left(-1,-\frac{1}{2}\right)$ implies a unit circle centered at $(0,0)$, $Re(z)\in\left(-\frac{1}{2},0\right)$ implies a unit circle centered at $(-1,0)$, $Re(z)=-\frac{1}{2}$ implies a line $Re(z)=\frac{1}{2}$ minus a segment bounded by the first two circles. At last, $Re(z)=-1,0$ are limit points when $n\to\infty$.

Now we should define the numerator R_n of α_{12}. Let

$$S_n = (p_1+p_2)^2(p_1^2+p_1p_2+p_2^2)^kT_n(p_1,p_2),\quad k=0,1,2 \tag{12}$$

Direct computation gives two variants

$$R_n(p_1,p_2) = S_n(-p_1,p_2) \tag{13}$$