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Effect of self-interaction on the phase diagram of a Gibbs-like measure derived by a reversible Probabilistic Cellular Automata

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A R T I C L E I N F O

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A B S T R A C T

Cellular Automata are discrete-time dynamical systems on a spatially extended discrete space which provide paradigmatic examples of nonlinear phenomena. Their stochastic generalizations, i.e., Probabilistic Cellular Automata (PCA), are discrete time Markov chains on lattice with finite single-cell states whose distinguishing feature is the *parallel* character of the updating rule. We study the ground states of the Hamiltonian and the low-temperature phase diagram of the related Gibbs measure naturally associated with a class of reversible PCA, called the *cross PCA*. In such a model the updating rule of a cell depends indeed only on the status of the five cells forming a cross centered at the original cell itself. In particular, it depends on the value of the center spin (*self-interaction*). The goal of the paper is that of investigating the role played by the self-interaction parameter in connection with the ground states of the Hamiltonian and the low-temperature phase diagram of the Gibbs measure associated with this particular PCA.

1. Introduction

Cellular Automata (CA) are discrete-time dynamical systems on a spatially extended discrete space. They are well known for – at the same time – being easy to define and implement and for exhibiting a rich and complex nonlinear behavior as emphasized for instance in [37,38] for CA on one-dimensional lattice. See [22] to precise the connections with the nonlinear physics. For the general theory of deterministic CA we refer to the recent paper [20] and references therein.

Probabilistic Cellular Automata (PCA) are CA straightforward generalization where the updating rule is stochastic. They inherit the computational power of CA and are used as models in a wide range of applications (see, for instance, the contributions in [32]). From a theoretic perspective, the main challenges concern the non-ergodicity of these dynamics for an infinite collection of interacting cells. Ergodicity means the non-dependence of the long-time behavior on the initial probability distribution and the convergence in law towards a unique stationary probability distribution (see [34] for details and references). Non-ergodicity is related to *critical phenomena* and it is sometimes referred to as *dynamical phase transition*.

Strong relations exist between PCA and the general equilibrium statistical mechanics framework [36,16,23]. Important issues are related to the interplay between disordered global states and ordered phases (*emergence of organized global states, phase transition*) [28]. Altough, PCA initial interest arose in the framework of Statistical Physics, in the recent literature many different applications of PCA have been proposed. In particular it is notable to remark that a natural context in which the PCA main ideas are of interest is that of evolutionary games [29–31].

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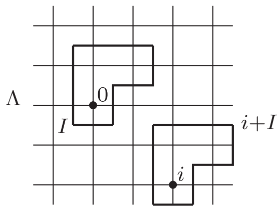

Fig. 1.1. Schematic representation of the action of the shift Θ_i defined in (1.1).

PCA dynamics are naturally defined on an infinite lattice. Given a local stochastic updating rule, one has to face the usual problems about the connections between the PCA dynamics on a finite subpart of the lattice and the dynamics on the infinite lattice. In particular, it was stated in [17] for translation-invariant infinite volume PCA with *positive rates*,[1] that the law of the trajectories, starting from any stationary translation-invariant distribution, is the Boltzmann–Gibbs distribution for some space–time associated potential. Thus phase transition for the space–time potential is intimately related to the PCA dynamical phase transition.

Moreover, see [14, Proposition 2.2], given a translation-invariant PCA dynamics, if there exists one translation-invariant stationary distribution which is a Gibbs measure with respect to some potential on the lattice, then all the associated translation-invariant stationary distributions are Gibbs with respect to the same potential.

In this paper we shall consider a particular class of PCA, called *reversible* PCA, which are reversible with respect to a Gibbs-like measure defined via a translation invariant multi-body potential. In this framework we shall study the zero and low-temperature phase diagram of such an equilibrium statistical mechanics-like system, whose phases are related to the stationary measures of the original PCA.

We shall now first briefly recall formally the definitions of Cellular Automata and Probabilistic Cellular Automata and then describe the main results of the paper.

1.1. Cellular Automata

Cellular Automata are defined via a local deterministic evolution rule. Let $\Lambda \subset \mathbb{Z}^d$ be a finite cube with periodic boundary conditions.

Associate with each site $i \in \Lambda$ (also called *cell*) the state variable $\sigma_i \in \mathcal{S}_0$, where $\mathcal{S}_0$ is a finite single-site space and denote by $\Omega := \mathcal{S}_0^\Lambda$ the *state space*. Any $\sigma \in \Omega$ is called a *state* or *configuration* of the system.

In order to define the evolution rule we consider I, a subset of the torus Λ, and a function $f_I : \mathcal{S}_0^I \to \mathcal{S}_0$ depending on the state variables in I. We also introduce the shift Θ_i on the torus, for any $i \in \Lambda$, defined as the map $\Theta_i : \Omega \to \Omega$

$$(\Theta_i\sigma)_j = \sigma_{i+j}. \tag{1.1}$$

The configuration σ at site j shifted by i is equal to the configuration at site $i+j$. For example (see Fig. 1.1) set $j = 0$, then the value of the spin at the origin 0 will be mapped to site i. The *Cellular Automaton* on Ω with rule f_I is the sequence $\sigma(0), \sigma(1), \ldots, \sigma(t), \ldots$ for t a positive integer, of states in Ω satisfying the following (deterministic) rule:

$$\sigma_i(t) = f_I(\Theta_i\sigma(t-1)) \tag{1.2}$$

for all $i \in \Lambda$ and $t \geq 1$.

Note the local and parallel character of the evolution: the value $\sigma_i(t+1)$, for all $i \in \Lambda$, of all the state variables at time $t+1$ depend on the value of the state variables at time t (parallel evolution) associated only with the sites in $i+I$ (locality).

1.2. Probabilistic Cellular Automata

The stochastic version of Cellular Automata is called *Probabilistic Cellular Automata* (PCA). We consider a probability distribution $f_\sigma : \mathcal{S}_0 \to [0,1]$ depending on the state σ restricted to I; we drop the dependence on I in the notation for future convenience. A Probabilistic Cellular Automata is the Markov chain $\sigma(0), \sigma(1), \ldots, \sigma(t), \ldots$ on Ω with transition matrix

$$p(\sigma, \eta) = \prod_{i \in \Lambda} f_{\Theta_i\sigma}(\eta_i) \tag{1.3}$$

for $\sigma, \eta \in \Omega$. We remark that f depends on $\Theta_i\sigma$ only via the neighborhood $i+I$. Note that, as in the deterministic case, the character of the evolution is local and parallel.

1.3. Description of the problem and results

Under suitable hypotheses on the probability distribution f_σ, for Λ finite, the Markov chain is irreducible and aperiodic, so that a unique stationary probability measure exists. On the other hand, irreducible and aperiodic PCA are in general not reversible. As already proven in [21,34,18] there exists a class of PCA which are reversible with respect to a Gibbs-like probability measure [14, Proposition 3.1] and, hence, they admit a sort of Hamiltonian. These models will be called *reversible PCA* (see [24, Section 3.5] for more details).

From the results in [14], see for instance Proposition 3.3 therein, it is possible to deduce that these Gibbs-like measures are either stationary or two-periodic for the PCA. Therefore it is quite natural to compare the behavior of these distributions to the one of the statistical mechanics counterpart.

Moreover, it is worth mentioning that also non-equilibrium properties of the PCA dynamics have been widely investigated. In [25], in the attractive reversible case and in absence of phase transition, the equivalence between an equilibrium weak-mixing condition and the convergence towards a unique equilibrium state with exponential speed was proven. In [1,8,7,9,27] the metastable behavior of a certain class of reversible PCA has been analyzed. In this framework the remarkable interest of a particular reversible PCA has been pointed out, called the cross PCA (see Section 3). It is a two-dimensional reversible PCA in which the updating rule of a cell depends on the status of the five cells forming a cross centered at the cell itself. In this model, the future state of the spin at a given cell

[1] A PCA is said to be with *positive rates* if the local updating rule is a distribution giving positive probability to any cell-state.