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# Fractal-based exponential distribution of urban density and self-affine fractal forms of cities

Yanguang Chen *, Jian Feng

*Department of Geography, College of Urban and Environmental Sciences, Peking University, Beijing 100871, PR China*

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ABSTRACT

Urban population density always follows the exponential distribution and can be described with Clark's model. Because of this, the spatial distribution of urban population used to be regarded as non-fractal pattern. However, Clark's model differs from the exponential function in mathematics because that urban population is distributed on the fractal support of landform and land-use form. By using mathematical transform and empirical evidence, we argue that there are self-affine scaling relations and local power laws behind the exponential distribution of urban density. The scale parameter of Clark's model indicating the characteristic radius of cities is not a real constant, but depends on the urban field we defined. So the exponential model suggests local fractal structure with two kinds of fractal parameters. The parameters can be used to characterize urban space filling, spatial correlation, self-affine properties, and self-organized evolution. The case study of the city of Hangzhou, China, is employed to verify the theoretical inference. Based on the empirical analysis, a three-ring model of cities is presented and a city is conceptually divided into three layers from core to periphery. The scaling region and non-scaling region appear alternately in the city. This model may be helpful for future urban studies and city planning.



## 1. Introduction

The mathematical models on urban density are special spatial correlation functions, which can be used to analyze spatial autocorrelation and spatio-temporal evolution of cities. Urban density distributions fall into two classes – the scale-free distribution without characteristic length, and the scale-dependent distribution with characteristic length. The former indicates the power-law distribution, while the latter mainly include the exponential distribution and the normal distribution. The power-law distribution usually suggests fractal structure, and the fractal dimension can be estimated with the number-radius scaling or box-counting method [4,9,19,33]. The common exponential distribution and normal distribution are not of fractal pattern, suggesting no fractional dimension. In practice, three kinds of distributions are always modeled by three functions: power function, exponential function, and normal function (Gaussian function).

Since Clark [12] employed the negative exponential function to describe the population density, more than eleven functions have been introduced to characterize urban density [4,5,7,28,45]. Among all these density models, three ones came to front successively. The first is the negative exponential function known as Clark's model, the second is the Gaussian function known as Sherratt–Tanner's model [34,37], and the third is the inverse power function known as Smeed's model [35]. The Clark model is well-known for geographers, representing the most influential form for urban density. The Sherratt–Tanner model has the advantage of Clark's model because of its simpler expression for mathematical analysis [13]. However, in empirical analysis of urban form, the negative exponential function gains an evident advantage over the Gaussian function. In terms of fractal concepts, Batty and Kim [3] argued forcibly that the use of Clark's model is fundamentally flawed due to its absence of a parameter

* Corresponding author.
*E-mail address:* chenyg@pku.edu.cn (Y. Chen).

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indicating space-filling competence of systems. They suggested that the most appropriate form for urban population density models is the inverse power function associated with fractal distribution rather than the negative exponential function indicative of the distribution with characteristic scale.

This paper will present a viewpoint differing to some extent from Batty and Kim [3]. We argue for the suggestion that the inverse power function is very significant for us to study urban density, but we argue against the opinion that Clark's model does not imply fractal structure and space filling. The inverse power function can be used to describe urban density of transport network, while the negative exponential function is suitable for characterizing urban population density in many cases. Therefore, the Clark model cannot always give place to the Smeed model. This paper will reinterpret Clark's model with the ideas from fractals. I will argue that the Clark model is not a simple exponential function, but the one indicating special exponential distribution based on fractal supports and denoting fractal form. Also this paper will reinterpret the claim of Parr [29,30], who suggested that the negative exponential function is more appropriate for describing population density in the urban area itself, while the inverse power function is more appropriate to the urban fringe and hinterland (see also [4]).

The innovation of the paper lies in following aspects. First, it will show how to understanding the self-affine fractal feature behind the exponential distribution by means of scaling analysis. Second, it will illuminate how to recognize the dimension of the urban density distributions with superficial characteristic scales. This is helpful for us to comprehend the nature of urban space. Third, based on the special exponential distribution with fractal properties, a three-ring city model is proposed for geographers and planners to grasp the spatial structure of cities. The three urban density models mentioned above oppose each other but also complement one another. This paper is devoted to researching into the exponential distribution, and at the same time, this work also discusses the normal distribution and power-law distribution for reference. The exponential model and the normal one can be formally unified in mathematical expression.

## 2. The dimension of the urban distribution with scale

### 2.1. Basic postulates, concepts and analytical methods

For simplicity, we only consider the monocentric cities with single core of growth. In many cases, a polycentric city can be treated as a monocentric one by changing coarse-graining level. Two postulates are put forward as follows. First, the *landform* is a fractal body, which influences urban land use and population distribution. Second, the urban *land use form* is a fractal pattern, and there exists an interaction between population distribution and land use structure. Both the landform and land use form compose the *physical infrastructure* of population distribution. The human aggregation is determined by the physical infrastructure and in turn reacts on it. The physical infrastructure can be regarded as a *fractal support*, on which a city grows and evolves.

If urban density satisfies the exponential distribution or normal distribution, it possesses a parameter indicating *characteristic length*. For example, in Clark's model, the relative rate at which the effect of distance attenuates used to be looked upon as this kind of parameter. The reciprocal of the *rate parameter* is a *scale parameter* indicating the *characteristic radius* ($r_0$) of urban population distribution [36]. It suggests some mean distance of human activities. In theory, the urban form which is similar to the "fractal dust" in appearance has no distinct boundary [39,40]. We can identify an urban boundary by a fractal approach [38], or the city clustering algorithm (CCA) [31]. The boundary forms an *urban envelope* [24]. The region within the envelope can be thought of as the area of a city ($A$). The radius of the circle with the same area as the urban area is termed *boundary radius* ($R_b = (A/\pi)^{1/2}$), representing the distance from a city center to its boundary.

If the characteristic radius of a city is a real constant independent of the city size considered, Clark's model is the conventional exponential function and has no singularity. A speculation is that the characteristic radius varies with the city size defined, and there is a scaling relation under dilation between the characteristic radius ($r_0$) and boundary radius ($R_b$). If so, Clark's model should be regarded as a special exponential function indicating self-affine fractal form, which will be validated in next section. The normal model can be handled in the same way. One of the keystones of this study is to illustrate this kind of scaling relation with observational evidence. The main analytical methods employed by this article include mathematical modeling and empirical analysis. The mathematical methods involve scaling analysis, spatial correlation analysis, and spectral analysis, which can be called "*3S analysis*" of cities. The 3S analysis is very effective to deal with the complicated mathematical models through simple ways.

As prearrangement, three fractal concepts should be made clear here. The first is *real fractal* (R-fractal), which needs no special explanation (see [4,18,27]. The second is pseudofractal (P-fractal), which suggests the fractional dimension coming from the non-fractal systems. This can be regarded as "fractal rabbits" [22]. The pseudofractals are always generated by the errors resulting from mathematical transformation and approximate treatment. The third is *quasifractal* (Q-fractal), which refers to such a case: intuitionally there is no fractal dimension, but empirically come out a fractional dimension that cannot be strictly distinguished from the real fractal dimension in practice. We have several approaches to demonstrating that the dimension of the common exponential distribution and normal distribution can be taken as $d = 2$. However, for the special exponential distribution and normal distribution, there exists a local fractal dimension.

### 2.2. Exponential distribution and self-affine fractal form

The special exponential model suggesting latent fractal nature can be derived from Clark's law. For the population density $\rho(r)$ at distance $r$ from the center of the city ($r = 0$), the exponential function can be expressed as