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Bifurcations of limit cycles in a quintic Lyapunov system with eleven parameters ☆

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ABSTRACT

In this paper, center conditions and bifurcation of limit cycles at the nilpotent critical point in a class of quintic polynomial differential system are investigated. With the help of computer algebra system MATHEMATICA, the first 12 quasi Lyapunov constants are deduced. As a result, sufficient and necessary conditions in order to have a center are obtained. The fact that there exist 12 small amplitude limit cycles created from the three order nilpotent critical point is also proved. Henceforth we give a lower bound of cyclicity of three-order nilpotent critical point for quintic Lyapunov systems, the result of Jiang et al. (2009) [18] was improved.

1. Introduction

The nilpotent center problem was theoretically solved by Stróżyna and Żołdek [14]. Nevertheless, in fact, given an analytic system with a monodromic point, it is very difficult to know if it is a focus or a center by their method because the computations are terrible and complicated, even in the case of polynomial systems of a given degree. In this paper, we consider an autonomous planar ordinary differential equation having a three-order nilpotent critical point with the form

$$\begin{aligned}\frac{dx}{dt}&=\mu y+a_{12}xy^2+a_{03}y^3+a_{31}x^3y+a_{22}x^2y^2-4b_{04}xy^3\\&\quad+a_{04}y^4+(1-\mu)x^2y-\mu y(x^2+y^2)^2,\\\frac{dy}{dt}&=-2\mu x^3+b_{21}x^2y+b_{03}y^3+b_{40}x^4-\frac{3}{2}a_{31}x^2y^2+b_{13}xy^3\\&\quad+b_{04}y^4+(\mu-1)xy^2+\mu x(x^2+y^2)^2,\end{aligned}\tag{1.1}$$

where $\mu\neq 0$, and all parameters are real.

In some suitable coordinates, the Lyapunov system with the origin as a nilpotent critical point can be written as

$$\begin{aligned}\frac{dx}{dt}&=y+\sum_{i+j=2}^{\infty}a_{ij}x^iy^j=X(x,y),\\\frac{dy}{dt}&=\sum_{i+j=2}^{\infty}b_{ij}x^iy^j=Y(x,y).\end{aligned}\tag{1.2}$$

Suppose that the function $y=y(x)$ satisfies $X(x,y)=0$, $y(0)=0$. Lyapunov proved (see for instance [3]) that the origin of system (1.2) is a monodromic critical point (i.e., a center or a focus) if and only if

$$\begin{aligned}&Y(x,y(x))=\alpha x^{2n+1}+o(x^{2n+1}),\quad \alpha<0,\\&\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial x}\right]_{y=y(x)}=\beta x^n+o(x^n),\\&\beta^2+4(n+1)\alpha<0,\end{aligned}\tag{1.3}$$

where n is a positive integer. The monodromy problem in this case was solved in [4] and the center problem in [12], see also in [14]. As far as we know there are essentially three differential ways of obtaining the Lyapunov constant: by using normal form theory [9], by computing the Poincaré return map [6] or by using Lyapunov functions [13]. Álvarez study the momodromy and stability for nilpotent critical points with the method of computing

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the Poincaré return map, see for instance [1]; Chavarriga study the local analytic integrability for nilpotent centers by using Lyapunov functions, see for instance [8]; Moussu study the center-focus problem of nilpotent critical points with the method of normal form theory, see for instance [12]. Takens proved in [15] that system (1.2) can be formally transformed into a generalized Liénard system. Álvarez proved in [2] that using a reparametrization of the time to simplify even more. Giacomini et al. in [16] prove that the analytic nilpotent systems with a center can be expressed as limit of systems non-degenerated with a center. therefore, any nilpotent center can be detected using the same methods that for a nondegenerate center, for instance the Poincaré–Lyapunov method can be used to find the nilpotent centers.

There are very few results known for concrete families of differential systems with monodromic nilpotent critical points. Gasull and Torregrosa in [10] have generalized the scheme of computation of the Lyapunov constants for systems of the form

$$\begin{aligned}\dot{x} &= y + \sum_{k\geqslant n+1} F_k(x,y),\\ \dot{y} &= -x^{2n-1} + \sum_{k\geqslant 2n} G_k(x,y),\end{aligned} \tag{1.4}$$

where F_k and G_k are $(1,n)$-quasi-homogeneous functions of degree k. Chavarriga, García, and Giné study the integrability of centers perturbed by (p,q)-quasi-homogeneous polynomials in [7].

For a given family of polynomial differential equations, let $N(n)$ be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree n. In [5] it is found that $N(3)\geqslant 2$, $N(5)\geqslant 5$, $N(7)\geqslant 9$; In [1] it is found that $N(3)\geqslant 3$, $N(5)\geqslant 5$; For a family of Kukles system with six parameters, in [2] it is found that $N(3)\geqslant 3$. In this paper. Recently, Liu and Li in [17] proved that $N(3)\geqslant 8$. Jiang et al. in [18] proved that $N(5)\geqslant 10$. In this paper, employing the integral factor method introduced in [11], we will prove $N(5)\geqslant 12$. To the best of our knowledge, our results on the lower bounds of cyclicity of three-order nilpotent critical points for quintic systems are new.

We will organize this paper as follows. In Section 2, we state some preliminary knowledge given in [11] which is useful throughout the paper. In Section 3, using the linear recursive formulae in [11] to do direct computation, we obtain with relative ease the first 12 quasi-Lyapunov constants and the sufficient and necessary conditions of center. This paper is ended with Section 4 in which the 12-order weak focus conditions and the fact that there exist 12 limit cycles in the neighborhood of the three-order nilpotent critical point are proved.

2. Preliminary knowledge

When the nilpotent critical point is a focus or center, it is more difficult to know whether or not it is a center, because in a neighborhood of the critical point, the method of the Poincaré formal series cannot be used in order to compute Lyapunov constants. But in [11] Liu and Li found that for the three-order nilpotent critical points always exist a formal inverse integrating factor, but it was not true for other order nilpotent critical points. They have shown for the three-order nilpotent critical point, a new definition of the focal values under the generalized triangle polar coordinates and the computation method of Lyapunov constants using the inverse integral factors. The ideas of this section come from [11], where the center-focus problem of three-order nilpotent critical points in the planar dynamical systems are studied. We first recall the related notions and results. For more details, please refer to [11].

The origin of system (1.2) is a three-order monodromic critical point if and only if the system could be written as the following real autonomous planar system

$$\begin{aligned}\frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^i y^j = X(x,y),\\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^i y^j = Y(x,y).\end{aligned} \tag{2.1}$$

Write that

$$X(x,y) = y + \sum_{k=2}^{\infty}\sum_{i+j=k} a_{ij}x^i y^j, \quad Y(x,y) = \sum_{k=2}^{\infty}\sum_{i+j=k} b_{ij}x^i y^j, \tag{2.2}$$

where for $k = 1, 2, \ldots$

By using the transformation of generalized polar coordinates

$$x = r\cos\theta, \quad y = r^2\sin\theta, \tag{2.3}$$

system (2.1) can become

$$\frac{dr}{d\theta} = \frac{-\cos\theta\left[\sin\theta(1-2\cos^2\theta) + \mu(\cos^2\theta + 2\sin^2\theta)\right]}{2(\cos^4\theta + \sin^2\theta)} r + o(r). \tag{2.4}$$

Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k \tag{2.5}$$

be a solution of (2.4) satisfying the initial condition $r|_{\theta=0} = h$, where h is small and

$$\begin{aligned} &v_1(\theta) = (\cos^4\theta + \sin^2\theta)^{\frac{-1}{4}} \exp\left(\frac{-\mu}{2}\arctan\frac{\sin\theta}{\cos^2\theta}\right),\\ &v_1(k\pi) = 1, \quad k = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{2.6}$$

Because for all sufficiently small r, we have $d\theta/dt < 0$. In a small neighborhood, we can define the successor function of system (2.1) as follows:

$$\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k=2}^{\infty} v_k(-2\pi)h^k. \tag{2.7}$$

We have the following result.

Lemma 2.1. *For any positive integer m, $v_{2m+1}(-2\pi)$ has the form*

$$v_{2m+1}(-2\pi) = \sum_{k=1}^{m} \zeta_k^{(m)} v_{2k}(-2\pi), \tag{2.8}$$

where $\zeta_k^{(m)}$ is a polynomial of $v_j(\pi)$, $v_j(2\pi)$, $v_j(-2\pi)$, $(j = 2, 3, \ldots, 2m)$ with rational coefficients.

It is different from the center-focus problem for the elementary critical points, we know from Lemma 2.1 that when $k > 1$ for the first non-zero $v_k(-2\pi)$, k is an even integer.