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# Approximate convex hull of affine iterated function system attractors

Anton Mishkinis [a,*], Christian Gentil [a], Sandrine Lanquetin [a], Dmitry Sokolov [b]

[a] Université de Bourgogne, LE2i, UMR 5158, Aile des Sciences de L'Ingénieur, Boite Postale 47870 9 Avenue Alain Savary, Dijon, Cedex 21000, France
[b] Université de Lorraine, LORIA, UMR 7503, Campus Scientifique, Boite Postale 239 Vandoeuvre-les-Nancy, Cedex 54506, France

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ABSTRACT

In this paper, we present an algorithm to construct an approximate convex hull of the attractors of an affine iterated function system (IFS). We construct a sequence of convex hull approximations for any required precision using the self-similarity property of the attractor in order to optimize calculations. Due to the affine properties of IFS transformations, the number of points considered in the construction is reduced. The time complexity of our algorithm is a *linear* function of the number of iterations and the number of points in the output approximate convex hull. The number of iterations and the execution time increases logarithmically with increasing accuracy. In addition, we introduce a method to simplify the approximate convex hull without loss of accuracy.



## 1. Introduction

Iterated function systems (IFS) define objects whose geometry can be very complex. This geometry is determined by a given set of transformations. An attractor may be evaluated by iterating these functions. Not only is this evaluation expensive, but the analysis and characterization (location, size) of the resulting shape can be complex. It is therefore interesting to have a more conventional form bounding the attractor.

The problem of bounding the IFS attractor occurs in many tasks, including numerical fractal analysis or the localization of an attractor. To guarantee the objects manufacturability, it is important to take into account the severe production constraints. So we must be able to quickly evaluate the approximation and localization of an attractor. The approximate convex hull may also be used to estimate normal vectors at points of an IFS fractal for real time realistic visualization.

One of the most challenging tasks in applications with dynamic virtual environments is fast and accurate collision detection. A typical environment is modeled by a collection of triangle meshes representing the scene geometry. Usually complex objects with fractal structure consist of a large number of triangles. Construction of the approximate convex hull will facilitate collision computations. In addition to accuracy, the approximate convex hull of various parts of the IFS attractor can be constructed.

In this paper, we demonstrate how the properties of an IFS may be exploited to compute convex hulls at any required accuracy. The article is organized in the following way: we start by recalling the basic concepts of an IFS and notations in Section 3. In Section 4, we examine each step of Martyn's approach [1] in order to generalize it to 3D and to optimize it. We show how to simplify the approximation of the convex hull, i.e., to reduce the number of points without losing accuracy. We then focus on the complexity of Martyn's approach and the complexity of our algorithm in Section 5. Finally, in Section 6, we compare results obtained with our algorithm and with Martyn's before concluding.

## 2. Related work

Methods to calculate approximations of the convex hull of an IFS attractor have already been developed.

Strichartz and Wang [2] and Kenyon et al. [3] studied the boundaries and the convex hulls of self-affine tiles that can be considered as the attractors of very special affine IFS, where all the transformations have the same linear part.

* Corresponding author. Tel.: +33 (6) 41684924.
*E-mail address:* mishkinis.anton@gmail.com (A. Mishkinis).

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Lawlor and Hart [4] presented an algorithm to construct a tight bounding polyhedron of the IFS attractor. An algorithm expresses the IFS-bounding problem as a set of linear constraints on a linear objective function, which can then be solved via standard techniques for linear convex optimization. This method works for a predetermined number of convex hull faces and shows the interactive rate only when this number is small.

More recently, Duda [5] and Martyn [1] have presented methods to calculate the approximate convex hull of the affine IFS attractor. The two methods are similar. The former is based on the so-called "width function" that returns the nearest to the point bounding half-space in a given direction. The latter is based on constructing a sequence of balls that bound corresponding parts of an attractor to approximate the convex hull. The approach is presented in 2D only.

Another important problem in computing the convex hull is to determine a bounding ball for the attractor of an IFS.

Gentil [6] described an approach based on the dichotomous search for the minimal radius of the ball that bounds the attractor of an IFS. The approach can be applied in a multi-dimensional space and calculates the result for a given precision.

Hart and DeFanti [7] introduced a method which starts with the unit ball centered at the origin. The algorithm iteratively produces a sequence of balls converging to the limit ball that bounds the attractor.

Rice [8] improved on Hart and DeFanti's approach by optimizing the radius of the bounding ball with the aid of a generic optimization package. He also showed that the center of the limit ball can be determined analytically by solving a system of linear equations.

Martyn [9] showed that the solution of this system is the centroid of the attractor with particular weights. To obtain a better approximation, he presented a heuristic iterative method called "balancing the attractor". The algorithm is not limited by the dimension of the space in which the attractor lies.

More recently, Martyn [10] presented a novel approach to approximate the smallest disc to enclose an affine IFS attractor at any accuracy. The method is based on a concept of spanning points he introduced to describe the extent of an IFS attractor.

In this article, we study an approximation of the convex hull of a given affine IFS attractor. This approximation will be given in the polytope form. Our model can be considered as a generalization and an optimization of Martyn's method. Our algorithm constructs a sequence of convex hull approximations using the self-similarity property of the attractor in order to reduce the number of necessary operations. In addition, we introduce a method to simplify the approximation of the convex hull without losing accuracy.

## 3. Background and notations

In this section we recall the major definitions and properties of iterated function systems as well as establish the notations used in this paper.

### 3.1. Iterated function system

Generally, an IFS is defined in a complete metric space $(\mathbb{X}, d)$, where $d$ is the associated metric. The transformation $T : \mathbb{X} \to \mathbb{X}$ is called contracting if and only if there exists a real $s$, $0 \leqslant s < 1$ such that $d(T(x),T(y)) < s \cdot d(x,y)$ for all $x, y \in \mathbb{X}$. The minimal coefficient $s$ which satisfies the inequality is called the contraction coefficient of the transformation $T$ with respect to the metric $d$.

We are substantially interested in attractors that are produced by an IFS composed of affine transformations. Each transformation can be described as follows: $T_i$: $x \mapsto Lx + b$, where $L$ is the linear part and $b$ is the vector of translation.

Atkins et al. proved [11] that affine IFS, for which the attractor exists, is hyperbolic. That means that there is a metric equivalent to the usual one so that each $T_i$ is contracting. Note that hyperbolic IFS is not necessarily contractive with respect to the usual euclidean metric.

In general case, it is a challenging task to determine contraction coefficients of the transformations. For our algorithm it is sufficient to provide the contraction coefficient majorants. Since all the metrics in finite dimensional space are equivalent, in the implementation we determine the contraction coefficients $s_i$ for the euclidean metric because of its simplicity of calculation. However, $s_i > 1$ does not imply that IFS does not have the attractor. In the cases where $\exists i$: $s_i > 1$ it is thus necessary to find another metric for which IFS is contractive and to determine the appropriate contraction coefficient majorants. This problem is beyond the scope of this article, for further informations see [11,12].

Thus, an affine IFS $\left(\mathbb{X}, \{T_i\}_{i=0}^{N-1}\right)$ consists of a finite set $\{T_0, \ldots, T_{N-1}\}$ of contracting affine transformations in a complete metric space $(\mathbb{X}, d)$. Let $\mathcal{H}(\mathbb{X})$ be the space of non-empty compact subsets of $\mathbb{X}$. Let $d_H$ be the Hausdorff distance induced by the metric $d$, i.e.:

$$d_H(A,B) = \max\{d(A,B), d(B,A)\},$$

where

$$d(A,B) = \max_{a\in A}\min_{b\in B} d(a,b).$$

Then $(\mathcal{H}(\mathbb{X}), d_H)$ is a complete metric space. The Hutchinson operator $\mathbb{T} : \mathcal{H}(\mathbb{X}) \to \mathcal{H}(\mathbb{X})$ associated with the IFS is defined by:

$$\mathbb{T}(K) = \bigcup_{i=0}^{N-1} T_i(K).$$

If $s_{max} = \max_{i=0,\ldots,N-1} s_i < 1$ then $\mathbb{T}$ is also contracting in the complete metric space $(\mathcal{H}(\mathbb{X}), d_H)$. According to Banach fixed point theorem [13], $\mathbb{T}$ has a unique fixed point $\mathcal{A}$. This fixed point is named the IFS attractor, i.e.:

$$\mathcal{A} = \bigcup_{i=0}^{N-1} T_i(\mathcal{A}). \tag{1}$$

The attractor of an IFS may be evaluated recursively. That is, it can be approximated by a sequence of objects $\{P_n\}_{n\in\mathbb{N}}$, which converges to $\mathcal{A}$. An initial element in the sequence defines by means of a primitive $P \in \mathcal{H}(\mathbb{X})$. The following elements are defined recursively: