Chaos, Solitons & Fractals 45 (2012) 765–771

Contents lists available at SciVerse ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Extending the D'alembert solution to space–time Modified Riemann–Liouville fractional wave equations

Cresus F.L. Godinho [a], J. Weberszpil [b,*], J.A. Helayël-Neto [c]

[a] Grupo de Física Teórica, Departamento de Física, Universidade Federal Rural do Rio de Janeiro, BR 465-07, 23890-971 Seropédica, RJ, Brazil
[b] Universidade Federal Rural do Rio de Janeiro, UFRRJ-IM/DTL, Av. Governador Roberto Silveira s/n, Nova Iguaçú, Rio de Janeiro, Brazil
[c] Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr. Xavier Sigaud 150, Urca, 22290-180 Rio de Janeiro, Brazil

ARTICLE INFO

Article history:
Received 22 December 2011
Accepted 13 February 2012
Available online 15 March 2012

ABSTRACT

In the realm of complexity, it is argued that adequate modeling of TeV-physics demands an approach based on fractal operators and fractional calculus (FC). Non-local theories and memory effects are connected to complexity and the FC. The non-differentiable nature of the microscopic dynamics may be connected with time scales. Based on the Modified Riemann–Liouville definition of fractional derivatives, we have worked out explicit solutions to a fractional wave equation with suitable initial conditions to carefully understand the time evolution of classical fields with a fractional dynamics. First, by considering space–time partial fractional derivatives of the same order in time and space, a generalized fractional D'alembertian is introduced and by means of a transformation of variables to light-cone coordinates, an explicit analytical solution is obtained. To address the situation of different orders in the time and space derivatives, we adopt different approaches, as it will become clear throughout this paper. Aspects connected to Lorentz symmetry are analyzed in both approaches.

1. Introduction

During the last decade the interest of physicists in non-local field theories has been steadily increasing. The main reason for this development is the expectation, that the use of these field theories will lead to a much more elegant and effective way of treating problems in particle and high-energy physics, as it is possible up to now with local field theories. A particular subgroup of non-local field theories is described with operators of fractional nature and is specified within the framework of fractional calculus. Fractional calculus provides us with a set of axioms and methods to extend the concept of a derivative operator from integer order n to arbitrary order α, where α is a real value. Problems involving fractional integrodifferential is an attractive framework that in recent years awakened the interest of some researchers in the study on the fractional dynamics in many fields of physics such as in anomalous diffusion [1], mechanics, engineering and other areas [2,3] and to deal with complex systems.

We can describe a complex system, as an 'open' system involving 'nonlinear interactions' among its subunits which can exhibit, under certain conditions, a marked degree of coherent or ordered behavior extending well beyond the scale or range of the individual subunits. Usually it consists of a large number of simple members, elements or agents, which interact with one another and with the environment, and which have the potential to generate qualitatively "new" collective behavior, the manifestations of this behavior being the spontaneous creation of new spatial, temporal, or functional structures. Within the realm of complexity, new questions in fundamental physics have been raised, which cannot be formulated adequately using traditional methods. Consequently a new research area has emerged, which allows for new insights and intriguing results using new methods and approaches.

* Corresponding author. Tel.: +55 021 2669 0817.
E-mail addresses: crgodinho@ufrrj.br (C.F.L. Godinho), josewebe@ufrrj.br, josewebe@gmail.com (J. Weberszpil), helayel@cbpf.br (J.A. Helayël-Neto).

0960-0779/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2012.02.008

The interest in fractional wave equations aroused in 2000, with a publication by Raspini [4]. He demonstrated that a 3-fold factorization of the Klein–Gordon equation leads to a fractional Dirac equation which contains fractional derivative operators of order $\alpha = 2/3$ and, furthermore, the resulting γ – matrices obey an extended Clifford algebra [5], let us quote that, in the same year, the integer case was also studied in the work of Ref. [6].

It is well known that most of physical systems might be described by Lagrangian functions and that frequently their dynamical variables are on first order derivatives. This occurs probably because classical and quantum theories with superior order derivatives do not present a lower limit to the energy. Besides, the quantum field theories containing superior derivatives of fields usually present states with negative norm or ghost states and consequently a costly price is payed with the lost of unitarity violation [7].

The quantum dynamics of a system can be described by operators acting on a state vector, this is a good reason to study operators algebras and its unfolds. The language of operators is very useful and important to the physicists, we know today that the quantum analog of derivatives like, for instance, $\partial/\partial t$ and $\partial/\partial x$ are operators. We still know that derivatives with respect to q_k and p_k can be represented by equations like Poisson brackets and its quantum analog are commutators involving operators. The modern theory of pseudo-differential operators took its shape in the sixties, but we can consider the thirties, because the quantization problem, preliminarily solved by Weyl, and since the 1980's this tool has also yielded many significant results in nonlinear partial differential equations PDE.

Recently, Jumarie [8] proposed a simple alternative definition to the Riemann–Liouville derivative. His Modified Riemann–Liouville derivative has the advantages of both the standard Riemann–Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non differentiable) functions and the fractional derivative of a constant is equal to zero. Jumarie's calculus seems to give a mathematical framework for dealing with dynamical systems defined in coarse-grained spaces and with coarse-grained time and, to this end, to use the fact that fractional calculus appears to be intimately related to fractal and self-similar functions.

We would like to stress that the choice of Jumarie's approach, besides the points already mentioned, is justified by the fact that chain and Leibinitz rules acquires a simpler form, which helps a great deal if changes of coordinates are performed. Moreover, causality seems to be more easily obeyed in a field-theoretical construction if we adopt Jumarie's approach.

As pointed out by Jumarie, non-differentiability and randomness are mutually related in their nature, in such a way that studies in fractals on the one hand and fractional Brownian motion on the other hand are often parallel in the same paper. A function which is continuous everywhere but is nowhere differentiable necessarily exhibits random-like or pseudo-random-features, in the sense that various samplings of this functions on the same given interval will be different. This may explain the huge amount of literature which extends the theory of stochastic differential equation to stochastic dynamics driven by fractional Brownian motion.

The most natural and direct way to question the classical framework of physics is to remark that in the space of our real world, the generic point is not infinitely small (or thin) but rather has a thickness. A coarse-grained space is a space in which the generic point is not infinitely thin, but rather has a thickness; and here this feature is modeled as a space in which the generic increment is not dx, but rather $(dx)^\alpha$ and likewise for coarse grained with respect to the time variable t.

It is noteworthy, at this stage, to highlight the interesting work by Nottale [11], where the notion of fractal space–time is first introduced.

In this work we claim that the use of an approach based on a sequential form of Jumarie's Modified Riemann-Liouvile [8] is adequate to describe the dynamics associated with fields theory and particles physics in the space of non-differentiable solution functions or in the coarse-grained space–time. Some possible realizations of fractional wave equations are given and the proposed solutions are analyzed. Based on this approach, we have worked out explicit solutions to a fractional wave equation with suitable initial conditions to carefully understand the time evolution of classical fields with a fractional dynamics. This has been pursued in $(1+1)$ dimensions where the adoption of the light-cone coordinates allow a very suitable factorization of the solution in terms of left-and-right-movers. First, by considering space–time partial fractional derivatives of the same order in time and space, a generalized fractional D'alembertian is introduced and by means of a transformation of variables to light-cone coordinates, an explicit analytical solution is obtained. Next, we address to the problem of different orders for time and space derivatives. In this situation, two different approaches have been adopted: one of them takes into account a non-differentiable space of solutions, whereas the other one considers a coarse-grained space–time as non-differentiable. For the former, there emerges an indicative of chiral symmetry violation. The latter points to a solution that depends on both the space and time orders of the derivatives. Aspects of Lorentz transform and invariance conditions are also analyzed. It is important to note that we are not assuming the validity of the semi-group property of the fractional derivatives [4] and are not working with generalized functions in the distributions sense [9] nor taking fractional power of operator [10]. Also, the solution technique here proposed does not make uses of Fourier transform and not necessarily Laplace transform. The work is organized as follows: After this Introduction, the Jumarie's modified fractional derivatives are briefly presented in Section 2. In the Section 3, the fractional wave equation is presented in the coarse-grained space–time and in non-differentiable function space of solutions. In Section 4 Lorentz transform and invariance conditions are analyzed. Following, in Section 5 we present an example. Finally, in Section 6, we cast our Concluding Comments. Two Appendices follow. The Appendices A and B treat respectively the aspects of calculation for different exponents and details of Lorentz transforms.