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Complexity of a kind of interval continuous self-map of finite type

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ABSTRACT

An interval map is called finitely typal, if the restriction of the map to non-wandering set is topologically conjugate with a subshift of finite type. In this paper, we prove that there exists an interval continuous self-map of finite type such that the Hausdorff dimension is an arbitrary number in the interval $(0,1)$, discuss various chaotic properties of the map and the relations between chaotic set and the set of recurrent points.

1. Introduction

It is well known that the discussion about dynamic behaviours of a system is often summarized as the study on an invariant set of the system [1–4]. However, it is an important method of studying problems that we establish topological conjugation or semi-conjugation relations between the system and the shift or the system and the subshift of finite type on symbolic space. Then according to the properties already found for dynamic system on symbolic space or the properties for the subshift of finite type we study the dynamic properties of some unknown systems.

Let (X,d) be a compact metric space, f: $X \to X$ be continuous, the topological entropy of f is denoted by $ent(f)$, whose definitions are as usual. f^n will denote the n-fold iterates of f.

A point $x \in X$ is called periodic point if there exists an n so that $f^n(x) = x$. The smallest positive integer n satisfying the above is called the prime period or least period of the point x. If every point in X is a periodic point with the same period n, then f is called periodic with period n. Denote the set of periodic points of f by $P(f)$.

$x \in X$ is called an almost periodic point of f, if for any $\varepsilon > 0$, $\exists N > 0$ such that for any $q \geqslant 0$, there exists r, $q < r \leqslant q + N$, we have $d(f^r(x),x) < \varepsilon$. Denote the set of almost periodic points of f by $A(f)$.

$x \in X$ is called a weakly almost periodic point of f, if for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that for any $n > 0$,

$$\#(\{r | f^r(x) \in V(x,\varepsilon),\ 0 \leqslant r < nN_\varepsilon\}) \geqslant n,$$

where $\#(\cdot)$ denotes the cardinal number of a set. Denote the set of weakly almost periodic points of f by $W(f)$.

$x \in X$ is called a recurrent point of f if the sequence $f(x), f^2(x), \ldots$ has a subsequence converging to x. The set of all recurrent points of f is denoted by $R(f)$.

$x \in X$ is called a non-wandering point of f if the sequence $f(x), f^2(x), \ldots$ has a subsequence converging to x. The set of all recurrent points of f is denoted by $R(f)$.

$x \in X$ is called an non-wandering point of f if for any $\varepsilon > 0$, there exists $n > 0$ such that

$$f^n(V(x,\varepsilon)) \cap V(x,\varepsilon) \neq \emptyset.$$

The set of all non-wandering points of f is denoted by $\Omega(f)$.

It is easy to see that

$$P(f) \subset A(f) \subset W(f) \subset R(f) \subset \overline{P(f)} \subset \Omega(f).$$

In this paper, based on [4] and [5], for any $s \in (0,1)$, we study the existence and chaotic properties of an interval continuous self-map of finite type whose Hausdorff dimension is s on a non-wandering set. In fact, we prove the following main theorem:

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The Main Theorem. *For any $s \in (0,1)$ there exists an interval continuous self-map of finite type f with the following properties:*

(1) *the Hausdorff dimension of $\Omega(f)$ is s*
(2) *$f|_{\Omega(f)}$ is Xiong chaos*
(3) *$f|_{\Omega(f)}$ is Devaney chaos and Wiggins chaos*
(4) *$f|_{\Omega(f)}$ is Kato chaos*
(5) *$f|_{\Omega(f)}$ is distributional chaos in a sequence*
(6) *$ent(f|_{\Omega(f)}) > 0$*
(7) *there exists an uncountable set in $A(f|_{\Omega(f)})$ such that $f|_{\Omega(f)}$ is distributional chaos*
(8) *there exists an uncountable set in $W(f|_{\Omega(f)}) - A(f|_{\Omega(f)})$ such that $f|_{\Omega(f)}$ is distributional chaos*
(9) *there exists an uncountable set in $R(f|_{\Omega(f)}) - W(f|_{\Omega(f)})$ such that $f|_{\Omega(f)}$ is distributional chaos*
(10) *$f|_{\Omega(f)}$ is two point distributional chaos*
(11) *$f|_{\Omega(f)}$ is Li–Yorke sensitivity.*

2. Basic concepts and lemmas

Let $E \subset X$ and $\delta > 0$, the diameter of subset E for the set X is denoted by $|E|$, i.e. $|E| = sup\{d(x,y) \mid x,y \in E\}$. $\{U_i\}$ is called the δ cover of the set E if $E \subset \bigcup_i U_i$ and $0 < |U_i| \leqslant \delta$ for every i. For any $0 \leqslant t < \infty$, let

$$H^t_\delta(E) = inf \sum_{i=1}^{\infty} |U_i|^t,$$

t dimension Hausdorff outer measure of E is defined as follows:

$$H^t(E) = \lim_{\delta \to 0^+} H^t_\delta(E)$$

if there exists a real number r satisfying with

$$H^t(E) = \begin{cases} \infty, & if\ 0 \leqslant t < r, \\ 0, & if\ r < t < \infty, \end{cases}$$

and such r is unique, the real number r is called the Hausdorff dimension of the set E and denoted by $dimE$.

Let $\Sigma = \{\alpha = \alpha_1\alpha_2\cdots|\alpha_i \in \{1,2\},\ \forall i \geqslant 1\}$ be one-sided symbolic space (with two symbols) and σ the shift on Σ. We define ρ: $\Sigma \times \Sigma \to R$ as follows: for any x, $y \in \Sigma$, if $x = x_1x_2\cdots$ and $y = y_1y_2\cdots$,

$$\rho(x,y) = \begin{cases} 0 & if \quad x = y, \\ \frac{1}{2^k} & if \quad x \neq y, \quad \text{and} \quad k = \min\{n|x_n \neq y_n\} - 1. \end{cases}$$

It is not difficult to check that ρ is a metric on Σ and the space (Σ, ρ) is compact. For an arbitrary tuple $B = b_1b_2\cdots b_n$ which is a finite arrangement of elements in $\{1,2\}$, $[B] = \{x = x_1x_2\cdots \in \Sigma, x_i = b_i, 1 \leqslant i \leqslant n\}$ is called a cylinder generated by B.

Let A be a square matrix of the second order whose elements are 0 or 1. The permutation of two symbols $\alpha_i\alpha_{i+1}$ is called to be admissible for A if the element lying in the α_i row and α_{i+1} column of A is 1. The sequence $\alpha = \alpha_1\alpha_2\cdots$ in Σ is called to be admissible for A if for each $i \geqslant 1$, $\alpha_i\alpha_{i+1}$ is admissible for A. We use Σ_A to denote the subset composed of all admissible sequences for A in Σ. It is easy to see that Σ_A is an invariant and closed set of σ. Hence, the restricted map $\sigma|_{\Sigma_A}$ is a subshift of σ, simplify, it is denoted by σ_A and called a subshift of finite type determined by A.

Let (X,f), (Y,g) be two dynamical system, if there exists a homeomorphic map h: $X \to Y$ such that $h(f(x)) = g(h(x))$ for any $x \in X$, we call that f is topologically conjugate with g and h is a topological conjugation from f to g.

f is said to be (topologically) transitive if for any pair of nonempty open sets U and V in X, there is a positive integer k such that

$$f^k(U) \cap V \neq \emptyset,$$

f is said to be (topological) mixing if for any pair of nonempty open sets U and V in X, there is a $T > 0$ such that for all $t \geqslant T$,

$$f^t(U) \cap V \neq \emptyset.$$

It is clear that a topological mixing system is topologically transitive.

Let X be a metric space. A mapping T: $X \to X$ is called contraction map if there exists a constant c with $0 \leqslant c < 1$ such that

$$d(T(x), T(y)) \leqslant c d(x,y), \quad x,y \in X.$$

The constant c is called the contractivity coefficient.

Lemma 2.1. *Let I be a closed interval, $\varphi_1, \varphi_2, \ldots, \varphi_m$ are all contraction maps on I, then there exists a unique nonempty compact set $E \subset I$ such that*

$$E = \varphi(E) = \bigcup_{i=1}^{m} \varphi_i(E),$$

where $\varphi = \bigcup_{i=1}^{m} \varphi_i$ is the transformation of the subset E of I. Moreover, for any nonempty compact set $F \subset I$, $\varphi^k(F)$ converges to E according to Hausdorff metric as $k \to \infty$.

Lemma 2.2. *Let I be a closed interval and $U \subset I$ an open interval. Suppose that the family of contraction maps $\varphi_1, \varphi_2, \ldots, \varphi_m$ on I satisfies with the conditions of open set to U, i.e.*

(1) $\varphi(U) = \bigcup_{i=1}^{m} \varphi_i(U) \subset U$,
(2) *$\varphi_1(U), \varphi_2(U), \ldots, \varphi_m(U)$ do not intersect each other.*

Moreover, suppose that for each i, there exist $0 < q_i < 1$, $0 < r_i < 1$ such that for any $x, y \in \overline{U}$,

$$q_i|x-y| \leqslant |\varphi_i(x) - \varphi_i(y)| \leqslant r_i|x-y|,$$

then $s \leqslant dimE \leqslant t$, where s and t are determined by the following equation:

$$\sum_{i=1}^{m} q_i^s = 1 = \sum_{i=1}^{m} r_i^t.$$

Lemma 2.1 and 2.2 come from Theorem 8.3 and 8.8 in [2] respectively.

Let f: $X \to X$ be continuous. A subset $Y \subset X$ is said to be Xiong-chaotic with respect to a given increasing sequence $\{m_i\}$ of positive integers if for any continuous map g: $Y \to X$ there is a subsequence $\{p_i\}$ of the sequence $\{m_i\}$ such that