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ScienceDirect
Chaos, Solitons and Fractals 40 (2009) 1574–1587
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

A study of biorthogonal multiple vector-valued wavelets

Jincang Han [a,b,*], Zhengxing Cheng [a], Qingjiang Chen [c]

[a] *School of Science, Xi'an Jiaotong University, Xi'an 710049, PR China*
[b] *School of Information, Lanzhou Commercial College, Lanzhou 730000, PR China*
[c] *School of Science, Xi'an University of Architecture and Technology, Xi'an 710055, PR China*

Accepted 10 September 2007

Communicated by Gerardo Iovane

Abstract

The notion of vector-valued multiresolution analysis is introduced and the concept of biorthogonal multiple vector-valued wavelets which are wavelets for vector fields, is introduced. It is proved that, like in the scalar and multiwavelet case, the existence of a pair of biorthogonal multiple vector-valued scaling functions guarantees the existence of a pair of biorthogonal multiple vector-valued wavelet functions. An algorithm for constructing a class of compactly supported biorthogonal multiple vector-valued wavelets is presented. Their properties are investigated by means of operator theory and algebra theory and time-frequency analysis method. Several biorthogonality formulas regarding these wavelet packets are obtained.

1. Introduction

It is well known that an interesting film of the XXI century is Matrix trilogy. In the final episode, the opposition between two forces represented by the architect, a man and a woman, the oracle becomes evident. The role of the architect is to maintain the order in the universe since in his view the beauty of the universe is in its rational and deterministic evolution. The function is the expression of the chaotic human behavior, that can change the evolution of the universe with its random contribution.

Under a new theory one usually expects, among other things, new differential equations or a variational principle in addition to special boundary and initial conditions. Above all the new theory should make new predictions as well as confirm older results, experimental as well as theoretical. By contrast, *E*-infinity theory [1] is more of a framework for understanding nature than just a new equation. What is the main point with *E*-infinity? El Naschie [2] state that the main point is stressing the fact that everything we see or measure is resolution dependent. In *E*-infinity view, spacetime is an infinite dimensional fractal that happens to have $D = 4$ as the expectation value for the topological dimension [3–5]. The topological value $3 + 1$ means that in our low energy resolution, the world appears to us as if it were four-dimensional. Observations of large scale structures show that the dimension changes if we consider different

* Corresponding author. Address: School of Science, Xi'an Jiaotong University, Xi'an 710049, PR China.
E-mail address: hanjincang@163.com (J. Han).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2007.09.037

energies, corresponding to different lengths-scale in universe [6]. Indeed, Mohamed EI Naschie with ε^{∞} has introduced a mathematical formulation to describe phenomena. His papers on *E*-infinity appears to be clearly a new framework for understanding and describing nature.

If we want to investigate the motion in the domain of frequencies, the mathematical tool is Fourier's transform. The context of *E*-infinity is quite different, since we need a transform that takes into account not only the frequencies but also the resolutions. The best candidate for this purpose is the wavelet transform. Indeed, engineers have discovered that it can be applied in all environments where the signal analysis is used. To implement the wavelet transform, we need to construct various wavelets. One of the objectives of this paper is to investigate the construction of wavelets.

Chen and Cheng [7] introduced the notion of vector-valued wavelets and studied the construction of orthogonal vector-valued wavelets. Bacchelli et al. [8] studied the existence of orthogonal multiple vector-valued wavelets by using the subdivision operator. Fowler and Li [9] implemented biorthogonal multiple vector-valued wavelet transforms by virtue of biorthogonal multiwavelets and employed them to study fluid flows in oceanography and aerodynamics. Therefore, studying vector-valued wavelets is useful in multiwavelets theory. However, multiwavelets [10,11] and vector-valued wavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms [12] but not necessary for discrete vector-valued wavelet transforms. Hence, it is necessary to investigate vector-valued wavelets. However, as yet there has not been a general method to obtain biorthogonal multiple vector-valued wavelets. Based on an observation in Chen and Cheng [7] and using blocking partition and full rank subdivision operator developed by Micchelli and Sauer [13] and some ideas from Bacchelli et al. [8], one of the purpose of this paper is to improve the result in [8], i.e., we obtain two main results which involve the existence and the construction of a class of compactly supported biorthogonal multiple vector-valued wavelets.

Wavelet packet theory has been applied to signal processing [14], image compression [15], and so on. Coifman and Meyer introduced the concept of orthogonal wavelet packets. Chui and Li [16] generalized the concept of orthogonal wavelet packets to the case of nonorthogonal wavelet packets so that wavelet packets can be applied to the spline wavelets and so on. Cohen and Daubechies [17] constructed biorthogonal wavelet packets which are more flexible in applications. Furthermore, Shen [18] constructed orthogonal nontensor multivariate wavelet packets. Inspired by [17–19], we give the definition of biorthogonal vector-valued wavelet packets and investigate their properties by means of operator theory and algebra theory and several biorthogonal formulas concerning the multiple vector-valued wavelet packets are obtained.

An outline of the paper is as follows: In Section 2, we introduce some notations used in the paper and briefly recall the concept of vector-valued multiresolution analysis. In Section 3, we discuss the existence of a pair of biorthogonal multiple vector-valued wavelet functions associated with a pair of biorthogonal multiple vector-valued scaling functions. In Section 4, we provide a constructive algorithm for constructing a class of compactly supported biorthogonal multiple vector-valued wavelets. In the final section, The notion of biorthogonal multiple vector-valued wavelet packets is introduced, and their properties will be investigated by means of operator theory and algebra theory.

2. Preliminaries and vector-valued multiresolution analysis

Let Z, R and C denote the collection of integers, real numbers and complex numbers, respectively. Set $Z_+ = \{\rho : \rho \in Z,\ \rho \geqslant 0\}$. Let $n, r, s \in Z_+$ be constants and $n \geqslant 2$. $\mathbf{I}_n$ and $\mathbf{O}$ stand for the $n \times n$ identity matrix and zero matrix, respectively. C^n stands for the n-dimensional complex Euclidean space. Let $\ell(Z)^n = \{\mathbf{c} : Z \to C^n\}$ denote the linear space of all s-vector-valued sequences and by $\ell^2(Z)^n$ the subspace of all sequences whose ℓ_2-norm satisfies $\|\mathbf{c}\|_2 := \left(\sum_{\iota=1}^{n}\sum_{k\in Z}|c_\iota(k)|^2\right)^{\frac{1}{2}} < +\infty$. Similarly, we define the linear space of all $n \times r$ matrix-valued sequences $\ell(Z)^{n\times r}$ and their subspace $\ell^2(Z)^{n\times r}$ as follow: $\ell(Z)^{n\times r} := \{B : Z \to C^{n\times r}\}$, and $\ell^2(Z)^{n\times r} := \{B : Z \to C^{n\times r}, \|B\|_2 = \left(\sum_{\iota=1}^{n}\sum_{\jmath=1}^{r}\sum_{k\in Z}|b_{\iota,\jmath}(k)|^2\right)^{\frac{1}{2}} < +\infty\}$. Moreover, we denote by $\ell_{00}(Z)$ the linear space of all finitely supported number sequences, a notion which can be trivially extended to a vector-valued sequence or a matrix-valued sequence. By

$$L^2(R, C^n) := \{\hbar(x) := (h_1(x), h_2(x), \ldots, h_n(x))^{\mathrm{T}}, h_\iota(x) \in L_2(R), \iota = 1, 2, \ldots, n\}, \tag{1}$$

we denote the space of square integrable vector-valued function where T means the transpose. For any $\hbar \in L^2(R, C^n)$, set $\hbar_{\iota;j,k} = 2^{j/2}\hbar_\iota(2^jx - k), \iota = 1, 2, \ldots, n, k, j \in Z$. For $\hbar \in L^2(R, C^n)$, $\|\hbar\|_2$ denotes the norm of vector-valued function $\hbar(x)$ as $\|\hbar\|_2 := \left(\sum_{\iota=1}^{n}\int_R |h_\iota(x)|^2 \mathrm{d}x\right)^{1/2}$, and its integration is defined as $\int_R \hbar(x)\mathrm{d}x := \left(\int_R h_1(x)\mathrm{d}x, \int_R h_2(x)\mathrm{d}x, \ldots, \int_R h_n(x)\mathrm{d}x\right)^{\mathrm{T}}$, and its Fourier transform is defined to be $\widehat{\hbar}(\omega) = \int_R \hbar(x)\mathrm{e}^{-\mathrm{i}x\omega}\mathrm{d}x$.