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# A four-wing attractor and its analysis

Guoyuan Qi [a,b,*], Barend Jacobus van Wyk [b], Michaël Antonie van Wyk [b]

[a] *Department of Automation, Tianjin University of Science and Technology, Tianjin 300222, China*
[b] *F'SATIE, Tshwane University of Technology, Pretoria 0001, South Africa*

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**Abstract**

In this paper, new properties of a four-dimensional chaotic system are investigated. These properties explain the behavior of the system and clarify why it can only generate two coexisting double-wing chaotic attractors but cannot produce a single four-wing chaotic attractor. It is shown that a new system with an extremely complex four-wing chaotic attractor and a larger positive Lyapunov exponent than the original system is formed by using these findings and introducing state feedback control to the system. Some basic dynamical behaviors and the complex structure of the new four-wing autonomous chaotic system are theoretically investigated. A detailed bifurcation analysis demonstrates the evolution process from local attractors to global attractors. The local attractors include two coexisting sinks, two coexisting single-wing periodic orbits and two coexisting double-wing chaotic attractors. The global attractors contain a diagonal double-wing periodic orbit, a diagonal double-wing chaotic attractor and a four-wing chaotic attractor. Spectral analysis indicates that the system in the four-wing chaotic mode has a very wide frequency bandwidth, confirming its random nature and its suitability to engineering applications such as secure communications.


## 1. Introduction

The proposals of new chaotic systems and the enhancement of existing chaotic attractors to generate more complex dynamics and topological structure are very important in the field of chaos theory and its application [1–3].

Generalizing Chua's circuit with multi-scroll attractors and generalizing the Lorenz system with a double-wing in a butterfly shape are two important efforts. There exist many examples of how to generate Chua's circuit with multi-scroll chaotic attractors. Several effective techniques have been developed and some simple circuits have been designed and implemented, including some generalized Chua's circuits and cellular neural networks [4–11]. Lü and Chen [12] surveyed the subject on generation of the multi-scroll chaotic attractors, including some fundamental theories, design methodologies, circuit implementations, and practical applications. Therefore, it is no longer a very difficult task to generalize Chua's circuit with multi-scroll attractors.

* Corresponding author. Address: Department of Automation, Tianjin University of Science and Technology, Tianjin 300222, China.

*E-mail addresses:* qi_gy@yahoo.com.cn, Guoyuanqi@gmail.com (G. Qi), vanwykb@tut.ac.za (B.J. van Wyk), vanwykma1@tut.ac.za (M.A. van Wyk).

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The generation of the Lorenz system has also been much investigated [13]. The Chen system, the generalized Lorenz system family, and the hyperbolic-type of generalized Lorenz canonical form [14–17] have been proposed. All these Lorenz-like systems are smooth with two quadratic terms, three equilibria, and produce a double-wing attractor.

The generation of autonomous chaotic systems with chaotic attractors with complicated topological structures including multi-wings, wider frequency bandwidths, and more complex dynamics and rich bifurcations is an important task. In such an attempt, a 3D system was proposed in [18]. At first, it was believed that this system could produce a four-wing chaotic attractor, termed a "four-scroll attractor" but it was later shown by the inventors to be a numerical artifact. It was shown not to be a real four-wing chaotic attractor but consisted of two coexisting and closely located double-wing attractors [15]. Another 3D autonomous quadratic chaotic system with five equilibria was designed in [19], which can truly produce a four-wing attractor under a constant-input control, but the frequency spectral bandwidth of the orbits produced by this system was very narrow.

Recently, we have found a new four-dimensional autonomous chaotic system, in which each equation contained a cubic term [20]. This system has very rich nonlinear dynamics, including chaos, period-doubling bifurcations, sinks, sources, etc. In particular it displays two coexisting symmetric single-wing chaotic attractors and two coexisting double-wing chaotic attractors [21]. A four-wing chaotic attractor was observed numerically which, but was also later shown to be an artifact by further theoretical analysis and by an analog circuit experiment. The observed four-wing attractor actually has two coexisting (upper and lower) double-wing attractors, which appeared simultaneously and were located arbitrarily closely in the phase space [22].

In this paper, some basic dynamical behaviors and the compound structure of the four-dimensional chaotic system found in [20] are reviewed and further investigated. A key property found is that the third and fourth equations in system (1) are exchangeable and symmetric. This property is the key to why the system can only generate two coexisting double-wing chaotic attractors, but cannot produce a single four-wing chaotic attractor. By adding a simple linear term to the system to cancel this property, a real four-wing attractor is generated. With the extra linear term, the new system has five different groups. Under certain conditions, the new system has a diagonal double-wing periodic orbit, and a diagonal double-wing chaotic attractor which can then be merged to form a single four-wing attractor by varying coefficient of the linear term. It is very significant to note that the new four-wing chaotic attractor has a more complicated topological structure and dynamics demonstrated by the larger positive Lyapunov exponent (LE) than the system in [20]. Spectral analysis shows that the system in the four-wing chaotic mode has a very broad frequency bandwidth, verifying its random nature, and indicating its suitability to engineering applications such as secure communications.

## 2. Basic properties of a 4D chaotic system

Qi et al. [20] proposed a 4D autonomous system with cubic nonlinearities, described by

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4, \\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4, \\
\dot{x}_3 &= -cx_3 + x_1x_2x_4, \\
\dot{x}_4 &= -dx_4 + x_1x_2x_3.
\end{aligned} \tag{1}$$

Here $x_i$ $(i = 1, 2, 3, 4)$ are the state variables and $a$, $b$, $c$, $d$ are positive real constants.

**Definition 2.1.** The equilibria are defined to be similar if they have the same group of eigenvalues. We consider the similar equilibria as one group.

In the following section basic properties of system (1) are firstly reviewed [21,22], followed by an investigation into the generation of new dynamics.

**Remark 2.1.** System (1) has nine real equilibria, i.e. $S_i$, it $\mathrm{i} = 0, 1, \ldots, 8$ which can be divided into three groups. The first group of equilibria is $S_{1,2,3,4}$, the second group of equilibria is $S_{5,6,7,8}$, and the third group is the origin.

**Remark 2.2.** System (1) is symmetric with respect to the $x_1$–$x_2$ and $x_3$–$x_4$ coordinate planes as well as the origin, respectively. Equilibria $S_{1,2}$, $S_{3,4}$, $S_{5,6}$, $S_{7,8}$ are all symmetric with respect to the $x_3$–$x_4$ plane, $S_{1,3}$, $S_{2,4}$, $S_{5,7}$, $S_{6,8}$ are all symmetric with respect to the $x_1$–$x_2$ plane, and $S_{1,4}$, $S_{2,3}$, $S_{5,8}$ and $S_{6,7}$ are all symmetric with respect to the origin.