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Some periodic and solitary travelling-wave solutions of the short-pulse equation

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Abstract

Exact periodic and solitary travelling-wave solutions of the short-pulse equation are derived.

1. Introduction

The short-pulse equation (SPE), namely

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1}$$

models the propagation of ultra-short light pulses in silica optical fibres [1].

In [2] it was shown that the SPE has a Lax pair that is of the Wadati–Konno–Ichikawa type (see [3], for example). Because of this result, it is not surprising that the SPE has a loop-soliton solution. This solution was found in [4] together with several other forms of solution.

In passing we note that there several other equations that have loop-soliton solutions; in [5] we gave a list of references in which some of these equations are discussed. We have presented two more such equations, namely the generalized Vakhnenko equation [6] and the modified generalized Vakhnenko equation [7].

In [2] it was shown that the SPE and the sine-Gordon equation (SGE) are equivalent to one another through a chain of transformations. In [4] various known solutions of the SGE were used to generate solutions to the SPE. The kink solution to the SGE leads to a travelling-wave solution of the SPE in the form of a loop soliton. (By a travelling-wave solution, we mean one in which the dependence on x and t is via the single variable $x - vt - x_0$, where v and x_0 are arbitrary constants.) The two-kink and kink–antikink solutions of the SGE also lead to multi-valued solutions of the SPE but they are not travelling waves. The breather solution to the SGE leads to a wave packet solution of the SPE. In the context of light pulses, the latter is the physically relevant solution.

The aim of the present paper is to complement the work in [4] by presenting other travelling-wave solutions to the SPE. The solution method is similar to the one that we have used previously to find periodic and solitary travelling-

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wave solutions to the Degasperis–Procesi equation [8], the Camassa–Holm equation [9] and the reduced Ostrovsky equation [10].

In Section 2 we derive an integrated form of the SPE. In Section 3 we find exact one-parameter travelling-wave solutions and classify them. In Section 4 we give our concluding remarks.

2. An integrated form of Eq. (1.1)

In order to seek travelling-wave solutions to the SPE it is convenient to introduce new variables z and η defined by

$$z := \frac{u}{|v|^{1/2}}, \quad \eta := \frac{x - vt - x_0}{|v|^{1/2}}, \tag{2.1}$$

where z is an implicit or explicit function of η, and $v \neq 0$. In this case (1.1) becomes

$$(z^2 + 2c)z_{\eta\eta} + 2z(1 + z_\eta^2) = 0, \quad \text{where} \quad c := \frac{v}{|v|} = \pm 1. \tag{2.2}$$

After one integration (2.2) gives

$$z_\zeta^2 = f(z), \tag{2.3}$$

where

$$f(z) := B^2 - (z^2 + 2c)^2, \tag{2.4}$$

B is a real positive constant and ζ is defined by

$$\frac{d\eta}{d\zeta} = z^2 + 2c. \tag{2.5}$$

We note that (2.2) is invariant under the transformation $z \to -z$. Also, when $c = -1$, (2.5) indicates that the solutions will have infinite slope when $z^2 = 2$.

3. Exact travelling-wave solutions of the SPE

For each choice of c, the possible types of travelling-wave solution of the SPE depend on the value, or range of values, of B. The types may be classified as follows:

3.1. $c = 1$, $B > 2$

In this case (2.4) may be written

$$f(z) := (a^2 + z^2)(b^2 - z^2), \quad \text{where} \quad a^2 = B + 2,\ b^2 = B - 2. \tag{3.1}$$

The bounded solutions to (2.2) are such that $-b \leqslant z \leqslant b$. By using results 213.00 and 310.02 from [11] to integrate (2.3) and (2.5), we find that these solutions are given in parametric form by

$$z = \pm\sqrt{\frac{4m}{(1 - 2m)}}\,\text{cn}\,(w|m), \quad \eta = \frac{[-w + 2E(w|m)]}{\sqrt{1 - 2m}}, \tag{3.2}$$

where $w(= 2\zeta/\sqrt{1 - 2m})$ is the parameter and

$$m = \frac{B - 2}{2B} \quad \text{so that} \quad 0 < m < \frac{1}{2}.$$

In (3.2) $\text{cn}(w|m)$ is a Jacobian elliptic function and the notation is as used in [12, Chapter 16]; $E(w|m)$ is the elliptic integral of the second kind and the notation is as used in [12, Section 17.2.8].

The solution (3.2) is a periodic hump or a periodic well corresponding respectively to the upper or lower choice of sign in (3.2). The wavelength of these solutions is

$$\lambda = \frac{4|-K(m) + 2E(m)|}{\sqrt{1 - 2m}}, \tag{3.3}$$