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An extended trace identity and applications

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Abstract

For the loop algebras in the form of non-square matrices, their commuting operations can be used to set up linear isospectral problems. In order to look for the Hamiltonian structures of the corresponding integrable evolution hierarchies of equations, an extended trace identity is obtained by means of commutators, which undoes the constraint on the known trace identity proposed by Tu [Guizhang Tu. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J Math Phys 1989;30(2):330–8], and has an obvious simplicity comparing with the quadratic-form identity given by Guo and Zhang [Fukui Guo, Yufeng Zhang. The quadratic-form identity for constructing the Hamiltonian structure of integrable systems. J Phys A 2005;38:8537–48] with the aspect of applications.

1. Introduction

In the section, we briefly recall the applications of the loop algebra $\widetilde{A}_1$. The necessity of improving the trace identity is proposed. Researching for integrable systems has been an important aspect in soliton theory [1]. Tu Guizhang [2] constructed the 2×2 loop algebra $\widetilde{A}_1$:

$$\widetilde{A}_1 = \{A = (a_{ij}(\lambda))_{2\times 2}, \quad a_{ij}(\lambda) = a_{ij}\lambda^m, \quad m = 0, \pm 1, \pm 2, \ldots, \mathrm{tr}(A) = 0\}$$

from which the isospectral Lax pairs could be established to further generate the zero curvature equation. It followed that the hierarchies of soliton equations were obtained, such as the AKNS hierarchy, the TA hierarchy, the BPT hierarchy and so on. Then their Hamiltonian structures were worked out by employing the trace identity [2,3]. In Refs. [4–6], a type of loop algebra $\widetilde{G}_0$ once was proposed which was used to get the expanding integrable systems, i.e., integrable couplings, of the AKNS hierarchy, the KN hierarchy, etc. However, their Hamiltonian structures could not be obtained by using the trace identity [2], it could not be reached that whether they were Liouville integrable or not. In another hand, a simple method for obtaining multi-component Lax integrable systems was presented with the help of the loop algebra $\widetilde{G}_1$ in the vector form in Ref. [5]. Since $\widetilde{G}_1$ is not presented in the square-matrix form, their Hamiltonian structures could not be obtained by use of the trace identity as well.

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In this paper, Section 2 presents two forms of the loop algebras and the trace identity as well as the quadratic-form identity. In Section 3, a kind of loop algebra $\widetilde{G}$ presented by the column vector form and a commutator are constructed, respectively. In Section 4, an extended trace identity under the framework of the loop algebra $\widetilde{G}$ is constructed. As its application, the Hamiltonian structure of an integrable system is exhibited.

2. Two types of loop algebras

We present two forms of the loop algebras. One is the square-matrix form $\widetilde{A}_{N-1}$, another is the s-dimensional column vector form $\widetilde{G}$. First, we introduce the loop algebra $\widetilde{A}_{N-1}$ and the resulting Lax pairs. $\widetilde{A}_{N-1}$ refers to

$$\widetilde{A}_{N-1} = \{A = (a_{ij}(\lambda))_{N\times N}, \quad a_{ij}\lambda^m, \quad m = 0, \pm 1, \pm 2, \ldots, \mathrm{tr}(A) = 0\}, \tag{1}$$

where the commuting operation is defined as $[A,B] = AB - BA, \quad A,B \in \widetilde{A}_{N-1}$. The linear isospectral problem related to $\widetilde{A}_{N-1}$ is given by

$$\begin{cases} \psi_x = U\psi, \quad U = e_0 + \sum_{i=1}^{p} u_i e_i, \quad \lambda_t = 0, \quad \psi = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}, \\ \psi_t = V\psi, \quad \{e_0, \ldots, e_p\} \subset \widetilde{A}_{N-1}, \quad u = (u_1, \ldots, u_p)^{\mathrm{T}} \end{cases} \tag{2}$$

whose compatibility condition is the zero curvature equation

$$U_t - V_x + [U,V] = 0. \tag{3}$$

The corresponding stationary zero curvature equation is that

$$V_x = [U,V]. \tag{4}$$

If the rank numbers of λ and $u_i (1 \leqslant i \leqslant p)$ can be defined, denoted by rank(λ), rank (u_i), respectively, such that $\mathrm{rank} e_0 = \mathrm{rank} u_i e_i = \alpha, 1 \leqslant i \leqslant p$, then U is called the same-rank, noted by

$$\mathrm{rank}(U) = \mathrm{rank}(\partial) = \alpha, \tag{5}$$

where $\partial = \frac{\partial}{\partial x}$. Take a solution of Eq. (4) to be $V = \sum_{m\geqslant 0} V_m \lambda^{-m}$, if rank($V_m$) can be defined such that $\mathrm{rank}(V_m\lambda^{-m}) = \eta = \mathrm{const.}$, $m \geqslant 0$, then we call V the same-rank, denoted by $\mathrm{rank}(V) = \eta$. Let two same-rank solutions V_1 and V_2 of Eq. (4) be linear dependent, i.e.

$$V_1 = \gamma V_2, \quad \gamma = \mathrm{const.}, \tag{6}$$

then we have

Theorem 1 ([1–3]). *If the relations* (5) and (6) *all hold, then*

$$\frac{\delta}{\delta u_i}\langle V, U_\lambda \rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left(\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle\right), \tag{7}$$

where V is a same-rank solution to Eq. (4), $\gamma = \mathrm{const.}, \quad \langle A,B\rangle = \mathrm{tr}(AB), \quad A,B \in \widetilde{A}_{N-1}$, *the identity* (7) *is called the trace identity, which has been proved in* Ref. [1].

Although we call the power series $V = \sum_{m\geqslant 0} V_m\lambda^{-m}$ the solution to Eq. (4), it is only required that the series makes the coefficients of the same-order powers of λ equal. Only part sum can be used in the integrable systems derived from Eq. (3) and some of the terms in V are required when using Eq. (7) to produce Hamiltonian structures. These facts are independent of the convergence or dispersion of V. That is why we do not discuss the convergence of V. The loop algebra $\widetilde{A}_1$ is often used for convenience, that is, take $\widetilde{A}_{N-1} = \widetilde{A}_1$ in Eqs. (2) and (7).

Let G be a s-dimensional Lie algebra with basis $e_1, e_2, \ldots, e_s$. Take $a = \sum_{k=1}^{s} a_k e_k, \ b = \sum_{k=1}^{s} b_k e_k \in G$. The commutator in G is given by $c = [a,b] = \sum_{k=1}^{s} c_k e_k$. The loop algebra $\widetilde{G}$ generated by G has the basis $e_k(m) = e_k\lambda^m, 1 \leqslant k \leqslant s, m \in \mathbf{Z}$(a integer set), the commuting operations read $[e_k(m), e_j(n)] = [e_k, e_j]\lambda^{m+n}$. The column vector form of $\widetilde{G}$ is given by

$$\widetilde{G} = \{a = (a_1, \ldots, a_s)^{\mathrm{T}}, \quad a_k = \sum_m a_{k,m}\lambda^m, \quad [a,b] = c = (c_1, \ldots, c_s)^{\mathrm{T}}\}. \tag{8}$$

The linear isospectral Lax pairs by using $\widetilde{G}$ can be taken as

$$\begin{cases} \psi_\partial = [U,\psi], \quad U, V, \psi \in \widetilde{G}, \\ \psi_t = [V,\psi], \quad \lambda_t = 0 \end{cases} \tag{9}$$