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Self-similar measures in multi-sector endogenous growth models

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ABSTRACT

We analyze two types of stochastic discrete time multi-sector endogenous growth models, namely a basic Uzawa–Lucas (1965, 1988) model and an extended three-sector version as in La Torre and Marsiglio (2010). As in the case of sustained growth the optimal dynamics of the state variables are not stationary, we focus on the dynamics of the capital ratio variables, and we show that, through appropriate log-transformations, they can be converted into affine iterated function systems converging to an invariant distribution supported on some (possibly fractal) compact set. This proves that also the steady state of endogenous growth models—*i.e.*, the stochastic balanced growth path equilibrium—might have a fractal nature. We also provide some sufficient conditions under which the associated self-similar measures turn out to be either singular or absolutely continuous (for the three-sector model we only consider the singularity).

1. Introduction

Almost thirty years after the seminal work of [7], it is now well known that also traditional (macro)economic models may give rise to complicated dynamics, including random dynamics eventually converging to (possibly singular) invariant measures supported on fractal sets. [32] borrowing from the iterated function system (IFS) literature [1,12,45] firstly show that standard stochastic economic growth models may show optimal dynamics defined by an IFS. The traditional one-sector growth model with Cobb–Douglas production and logarithmic utility has been extensively studied later. [28] shows that its optimal path converges to a singular measure supported on a Cantor set, characterizing singularity versus absolute continuity of the invariant probability in terms of the parameters' values. [29–31] further generalize the model and provide also an estimate of the Lipschitz constant for the maps of the optimal policy defined by an IFS.[1] Only recently, the analysis has been extended in order to consider two-sector growth models, nowadays predominant in economic growth theory. [17] show that in a two-sector model with physical and human capital accumulation the optimal dynamics for the state variables can be converted through an appropriate log-transformation into an IFS converging to an invariant measure supported on a generalized Sierpinski gasket.

The aim of this paper it to further extend the analysis of fractal outcomes in optimal economic growth models by

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[1] Other recent applications of the IFS theory showing that some economic growth model converge to an invariant distribution supported on a Cantor set are [24] and [39]. Specifically, [25] analyzes a two-sector Solow model, while [39] consider the sustainability problem in a stochastic economy–environment model.

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studying the behavior of multi-sector endogenous growth models. Indeed, thus far the focus has always been placed on neoclassical growth models, in which at steady state the economic growth rate is null, and nothing has been said on whether also a perpetually growing economy (*i.e.*, an economy experiencing a strictly positive steady state growth rate) may achieve a fractal-type steady state. We thus analyze two alternative models of endogenous growth, specifically a two-sector and a three-sector model, based on the Uzawa–Lucas [23,44] and [16] models, respectively. We show that even whenever perpetual growth is admissible the economy may develop along a (stochastic) balanced growth path equilibrium characterized by a fractal nature. However, since in such a framework the optimal dynamics of (physical, human and technological) capital are not stationary, we consider the dynamics of the capital ratio variables (specifically, the physical to human capital and technological to human capital ratios) and show that, through appropriate log-transformations, they can be converted into affine IFS converging to some distribution supported on a compact set which may be a fractal.[2] We then also provide some sufficient conditions under which the associated self-similar measures turn out to be singular and absolutely continuous.

The paper proceeds as follows. In Section 2 the main results from the IFS theory that we will need in our analysis are briefly recalled and novel sufficient conditions (Theorem 5) for singularity of the invariant distribution are provided for a special class of two-dimensional affine IFS. In Section 3 we consider the simplest form of multi-sector endogenous growth models, namely a Uzawa–Lucas [23,44] model driven by human capital accumulation. In Section 4 we analyze an extended version of the model, that is a three-sector model, as in [16], in which human capital is endogenously allocated across three (physical, human and knowledge) sectors. For both the models, we derive the optimal dynamics and construct an affine IFS conjugate to the optimal dynamics of stationary variables (the physical to human capital, and, in the latter, also the knowledge to human capital ratios). We provide, directly in terms of parameters of the model, sufficient conditions for the attractor of this conjugate IFS to be a fractal set (the Cantor set for the two-sector model and a generalized Sierpinski gasket for the three-sector model). We also identify sufficient conditions under which the self-similar measures turn out to be singular and absolutely continuous. In Section 5 we build some examples of attractors, while Section 6 presents concluding remarks and proposes directions for future research.

2. Iterated function systems

An Iterated Function System (IFS) is a finite collection of contractive maps which are defined on a complete metric space. This collection of maps allows to formalize the notion of self-similarity and the definition of invariant set or attractor of the IFS. An Iterated Function System with Probabilities (IFSP), instead, consists of the above collection of IFS maps together with an associated set of probabilities. An IFSP induces a Markov operator on the set of all Borel probability measures and a notion of self-similar invariant measure. More details on these can be found in the fundamental works by [2,12]. Applications of these methods are in image compression, approximation theory, signal analysis, denoising, and density estimation [11,13–15,18–22,26,27]. Now we recall, without proofs, some well known basic properties that will be used in the next sections.

We briefly introduce the notion of Hausdorff measure and Hausdorff dimension (more details can be found in [1]). Let (X, d) be a metric space and let diam (E) denote the diameter of a subset E of X. Let $s \geq 0$ and $\delta > 0$, and define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{k=1}^{\infty}[\operatorname{diam}(E_k)]^s : E \subset \bigcup_{i=1}^{\infty} E_k, \operatorname{diam}(E_k) < \delta\right\}.$$

Now let us define

$$\mathcal{H}^s(E) = \sup_{\delta>0}\mathcal{H}^s_\delta(E) \tag{1}$$

Definition 1. $\mathcal{H}^s(E)$ in (1) is called the s-dimensional Hausdorff measure. Furthermore, there exists a unique number $s_0 \geq 0$ such that $\mathcal{H}^s(E) = \infty$ for $0 \leq s < s_0$ and $\mathcal{H}^s(E) = 0$ for $s > s_0$. The number s_0 is called the *Hausdorff dimension* of E and it is denoted by $\dim_H(E)$.

In what follows, let (X, d) be a complete metric space and $w = \{w_0, \ldots, w_{m-1}\}$ a set of m injective contraction maps w_i: $X \to X$, to be referred to as an *m-map IFS*. Let $0 < \lambda_i < 1$ denote the contraction factors of w_i and define $\lambda := \max_{i\in\{0,\ldots,m-1\}}\lambda_i$; clearly $0 < \lambda < 1$. Associated with the IFS mappings $w_0, \ldots, w_{m-1}$ there is a set-valued mapping $\hat{w} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined over the space $\mathcal{K}(X)$ of all non-empty compact sets in X as

$$\hat{w}(A) := \bigcup_{i=0}^{m-1} w_i(A), \qquad \forall A \in \mathcal{K}(X), \tag{2}$$

where $w_i(A) = \{w_i(x) : x \in A\}$ is the image of A under w_i, $i = 0, 1, \ldots, m-1$. Let $\hat{w}^t(A) = \hat{w}\left[\hat{w}^{t-1}(A)\right]$ for all $t \geq 1$, with $\hat{w}^0(A) = A$. The *Hausdorff distance* d_H is defined as

$$d_H(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{x\in B}\inf_{y\in A} d(x, y)\right\},$$
$$\forall A, B \in \mathcal{K}(X).$$

Theorem 1 ([12]). *$(\mathcal{K}(X), d_H)$ is a complete metric space and $\hat{w}$ is a contraction mapping on $(\mathcal{K}(X), d_H)$:*

$$d_H\left(\hat{w}(A), \hat{w}(B)\right) \leq \lambda d_H(A, B), \quad \forall A, B \in \mathcal{K}(X).$$

Therefore, there exists a unique set $A^* \in \mathcal{K}(X)$, such that $\hat{w}(A^*) = A^*$, the so-called attractor (or *invariant set*) *of the IFS $\hat{w}$. Moreover, for any $A \in \mathcal{K}(X)$, $d_H\left(\hat{w}^t(A), A^*\right) \to 0$ as $t \to \infty$.*

2.1. Invariant measures

Let $p = (p_0, p_1, \ldots, p_{m-1})$, $0 < p_i < 1$, $i = 0, 1, \ldots, m-1$, be a partition of unity associated with the IFS mappings

[2] The advantage of introducing such a log-transformation consists of obtaining an affine system topologically conjugate to the original nonlinear system which allows to exploit the mathematical theory on IFS, thus simplifying the characterization of existence and uniqueness of (stochastic) fixed points. Without such a transformation, we would need to rely on more cumbersome ad-hoc approaches, like analyzing the eventual monotonicity properties of the optimal policies and dynamics, as, *e.g.*, [8] did in their seminal work.