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Dynamics of a Bertrand duopoly with differentiated products and nonlinear costs: Analysis, comparisons and new evidences

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ABSTRACT

This paper studies mathematical properties and dynamics of a duopoly with price competition and horizontal product differentiation by introducing quadratic production costs (decreasing returns to scale), thus extending the model with linear costs (constant returns to scale) of Fanti et al. [11]. The economy is described by a two-dimensional non-invertible discrete time dynamic system. The paper first determines fixed points and other invariant sets, showing that synchronized dynamics can occur. Then, stability properties are compared in the cases of quadratic costs and linear costs by considering the degree of product differentiation and the speed of adjustment of prices as key parameters. It is also shown that synchronization takes place if products tend to be relatively complements and stressed similarities and differences between models with quadratic and linear costs. Finally, the paper focuses on the phenomenon of multistability thus underlying new evidences in comparison with the model with linear costs.

1. Introduction

This paper extends the work of Fanti et al. [11] by considering a nonlinear duopoly with price competition, horizontal differentiation and quadratic production costs (decreasing returns to scale) instead of linear costs (constant returns to scale).

The analysis of economic models with product differentiation has become of greater importance in the oligopoly literature – especially in the case of price-setting firms – both in the absence of managerial delegation contracts [16] and when ownership and management are separate [13].

The study of nonlinear dynamics in oligopolies [8] has essentially concentrated on the case of quantity-setting firms to analyze long-term outcomes when information is complete [14] or incomplete [6,7], while leaving the case of price-setting firms treated only partially. Indeed, some exceptions actually exist in this direction. Specifically, Ahmed et al. [1] have taken into account a multi-team Bertrand game where the dynamics of the economy is studied by considering the ”Puu's incomplete information dynamical system” [1], p. 1182, that followed the work of Puu [15]. Subsequently, Zhang et al. [17] have studied the local properties of a repeated duopoly model with price competition, linear demand, linear costs and incomplete information of players, while Fanti et al. [11] have revisited their work by pointing out microeconomic foundations and the global analysis. In a recent work, Ahmed et al. [2] have dealt with the local properties of a model with linear costs and CES preferences.

The aim of this work is to deepen the study of the behaviour of nonlinear duopolies with price competition by considering that firms operate with a decreasing returns-to-scale technology, incomplete information and linear market demand (quadratic preferences). The assumption of

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incomplete information follows the literature led by Bischi et al. [7]. In this case, in fact, firms adopt specific behavioural rules (such as adjustment mechanisms based on marginal profits or the so called *Local Monopolistic Approximation*, as in [6]) in the product market.

The present paper stresses the mathematical properties of a duopoly with price competition with nonlinear production costs and then outlines similarities and differences with respect to the model with linear costs. When the cost function is nonlinear we find different dynamic outcomes than when firms operate with a linear cost function. Specifically, we find that synchronized dynamics increases in complexity by starting from the case of independent products and moving toward complementary or substitutability. In the case of independent products (i.e., each single firm behaves just like a monopolist in the market), results are different depending on the extent of market demand. In fact, the flip bifurcation for which the Nash equilibrium loses stability occurs earlier when costs are quadratic (resp. linear) if the extent of market demand is large (resp. small).

With regard to multistability, there exist coexisting attractors in either cases of complementarity and substitutability. However, when products are complements the structure of the basins of attraction may result to be complex, so that the final outcome of the economy may be unpredictable. The difference in terms of policy insights is then clear in the two cases. When products are substitutes, it is possible to provide adequate policies on the degree of product substitutability (for instance, advertising investments) to drive the economy toward the target, while when products are complements policies may not be effective or have detrimental effects.

The rest of the paper proceeds as follows. Section 2 sets up a duopoly with price competition, linear demand and decreasing returns to scale. It also outlines the two-dimensional map that characterizes the evolution of prices from one period to another. Section 3 determines the fixed points and other invariant sets. Then, local stability is investigated and a comparison with the dynamics produced by the model with linear costs is also presented. Similar with the case of linear costs, Section 4 shows that synchronization is likely to emerge if products are complements. In addition, the phenomenon of multistability is investigated, thus underlying new evidences. Specifically, in the particular case of independent products the paper shows that the size of the extent of market demand is crucial to determine whether the primary period doubling bifurcation occurs earlier in the case of quadratic or linear costs. Section 5 outlines the conclusions.

2. The model

The economy is composed of firms and consumers. There exists a competitive sector that produces the numeraire good $k \geq 0$ (whose price is normalized to 1), and a duopolistic sector where firm 1 and firm 2 produce (horizontally) differentiated products of variety 1 and variety 2, respectively. Let $p_i \geq 0$ and $q_i \geq 0$ be the price and quantity of product of firm i $(i = 1, 2)$, respectively.

Consumers. There exists a continuum of identical consumers that have preferences toward goods q_1, q_2 and k described by the separable utility function $V(q_1, q_2, k) : \mathbb{R}^3_+ \to \mathbb{R}_+$ and specified by $V(q_1, q_2, k) = U(q_1, q_2) + k$, where $U(q_1, q_2) : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a twice differentiable function. The representative consumer aims at maximising $V(q_1, q_2, k)$ subject to $p_1q_1 + p_2q_2 + k = M$ (budget constraint), where $M > 0$ is the exogenous nominal income of the consumer. This income is high enough to avoid the existence of corner solutions. In addition, there are no income effects on the duopolistic sector. The consumer's optimization problem is $\max_{\{q_1,q_2\}} U(q_1, q_2) - p_1q_1 - p_2q_2 + M$. We assume that consumers' preferences toward q_1 and q_2 are captured by the utility function:

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}\left(q_1^2 + q_2^2 + 2dq_1q_2\right), \tag{1}$$

where $a > 0$ is the extent of market demand of both goods and $-1 < d < 1$ is the degree of horizontal product differentiation. If $d = 0$ products of variety 1 and variety 2 are independent and each firm behaves as a monopolist. If $d > 0$ (resp. $d < 0$) products are substitutes (resp. complements), while when $d \to 1$ (resp. $d \to -1$) they tend to be perfect substitutes (resp. perfect complements). By using (1), the consumer's maximization procedure gives the following inverse demands of good 1 and good 2, respectively:

$$p_1 = a - q_1 - dq_2 \quad \text{and} \quad p_2 = a - q_2 - dq_1. \tag{2}$$

From (2) the corresponding direct demands are then given by:

$$q_1 = \frac{a(1-d)-p_1+dp_2}{1-d^2} \quad \text{and} \quad q_2 = \frac{a(1-d) - p_2 + dp_1}{1-d^2}. \tag{3}$$

Duopolistic firms. We assume that firm *i* produces with the decreasing returns-to-scale technology $q_i = \sqrt{L_i}$, where L_i is the labour force employed. Firm *i*'s cost function is $c_i = wL_i$, where $w > 0$ is the cost per unit of labor. The cost function can then be written as $c_i = wq_i^2$, so that average and marginal costs are respectively given by wq_i and $2wq_i$, i.e. marginal costs are higher than average costs for every $q_i > 0$.

Firm *i* maximizes profits $\Pi_i = p_iq_i - wq_i^2$ with respect to p_i. Then, by using (3) marginal profits of *i*th firm are given by

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{[a(1-d)+dp_j](1-d^2+2w) - 2(1-d^2+w)p_i}{(1-d^2)^2}, \quad i, j = 1, 2, \quad i \neq j. \tag{4}$$

Dynamic setting. Consider now a dynamic setting where time is indexed by $t \in \mathbb{Z}_+$. By following Bischi et al. [7] and Fanti et al. [10,11], we assume that both players have limited information. In order to set the price between two subsequent periods, both firms follow an adjustment process based on local estimates of their own marginal profit in the current period. This is given by:

$$p_{i,t+1} = p_{i,t} + \alpha p_{i,t}\frac{\partial \Pi_i(p_{i,t}, p_{j,t})}{\partial p_{i,t}}, \quad i = 1, 2, \quad t \in \mathbb{Z}_+, \tag{5}$$

where $\alpha > 0$ is the speed of adjustment and $\frac{\partial \Pi_i}{\partial p_i}$ is determined by (4). Equation (5) is the adjustment mechanism of prices in the case of Bertrand competition of the corresponding rule for quantities in Cournot oligopolies. Indeed, this adjustment mechanism represents a natural extension of the adjustment mechanism of quantities proposed by