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Multi-component generalization of the Camassa–Holm equation

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ABSTRACT

In this paper, we propose a multi-component system of the Camassa–Holm equation, denoted by CH(N, H), with $2N$ components and an arbitrary smooth function H. This system is shown to admit Lax pair and infinitely many conservation laws. We particularly study the case $N = 2$ and derive the bi-Hamiltonian structures and peaked soliton (peakon) solutions for some examples.

1. Introduction

In 1993, Camassa and Holm derived the well-known Camassa–Holm (CH) equation [1]

$$m_t + 2mu_x + m_x u = 0, \quad m = u - u_{xx} + k, \tag{1}$$

(with k being an arbitrary constant) with the aid of an asymptotic approximation to the Hamiltonian of the Green–Naghdi equations. Since the work of Camassa and Holm [1], more diverse studies on this equation have remarkably been developed [2–12]. The most interesting feature of the CH equation (1) is that it admits peakon solutions in the case $k = 0$. The stability and interaction of peakons were discussed in several references [13–17]. In addition to the CH equation, other similar integrable models with peakon solutions were found [18,19]. Recently, there are two integrable peakon equations found with cubic nonlinearity. They are the following cubic equation [3,20–22]

$$m_t + \frac{1}{2}\left[m(u^2 - u_x^2)\right]_x = 0, \quad m = u - u_{xx}, \tag{2}$$

and the Novikov's equation [23,24]

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}. \tag{3}$$

There is also much attention in studying integrable multi-component peakon equations. For example, in [25–28], multi-component generalizations of the CH equation were derived from different points of view, and in [29], multi-component extensions of the cubic nonlinear equation (2) were studied.

In a previous paper [30], we proposed a two-component generalization of the CH equation (1) and the cubic nonlinear CH equation (2)

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$$\begin{cases} m_t = (mH)_x + mH - \frac{1}{2}m(u-u_x)(v+v_x), \\ n_t = (nH)_x - nH + \frac{1}{2}n(u-u_x)(v+v_x), \\ m = u - u_{xx}, \qquad n = v - v_{xx}, \end{cases} \tag{4}$$

where H is an arbitrary smooth function of u, v, and their derivatives. Such a system is interesting, since different choices of H lead to different peakon equations. We presented the Lax pair and infinitely many conservation laws of the system for the general H, and discussed the bi-Hamiltonian structures and peakon solutions of the system for the special choices of H. In [31], Li, Liu and Popowicz proposed a four-component peakon equation which also contains an arbitrary function. They derived the Lax pair and infinite conservation laws for their four-component equation, and presented a bi-Hamiltonian structure for the equation in the special case that the arbitrary function is taken to be zero.

In this paper, we propose the following multi-component system

$$\begin{cases} m_{j,t} = (m_jH)_x + m_jH + \frac{1}{(N+1)^2}\sum_{i=1}^{N}[m_i(u_j-u_{j,x})(v_i+v_{i,x}) + m_j(u_i-u_{i,x})(v_i+v_{i,x})], \\ n_{j,t} = (n_jH)_x - n_jH - \frac{1}{(N+1)^2}\sum_{i=1}^{N}[n_i(u_i-u_{i,x})(v_j+v_{j,x}) + n_j(u_i-u_{i,x})(v_i+v_{i,x})], \\ m_j = u_j - u_{j,xx}, \qquad n_j = v_j - v_{j,xx}, \quad 1 \le j \le N, \end{cases} \tag{5}$$

where H is an arbitrary smooth function of u_j, v_j, $1 \le j \le N$, and their derivatives. The system contains $2N$ components and an arbitrary function H. For $N = 1$, this system becomes the two-component system (4). Therefore, system (5) is a kind of multi-component generalization of the two-component system (4). Due to the presence of the function H, system (5) is actually a large class of multi-component equations. We show that the multi-component system (5) admits Lax representation and infinitely many conservation laws. Although having not found a unified bi-Hamiltonian structure of the system (5) for the general H yet, we demonstrate that for some special choices of H, one may find the corresponding bi-Hamiltonian structures. As examples, we derive the peakon solutions in the case $N = 2$. In particular, we obtain a new integrable model which admits stationary peakon solutions.

The whole paper is organized as follows. In Section 2, the Lax pair and conservation laws of Eq. (5) are presented. In Section 3, the Hamiltonian structures and peakon solutions of Eq. (5) in the case $N = 2$ are discussed. Some conclusions and open problems are addressed in Section 4.

2. Lax pair and conservation laws

We first introduce the N-component vector potentials $\vec{u}$, $\vec{v}$ and $\vec{m}$, $\vec{n}$ as

$$\begin{aligned} &\vec{u} = (u_1, u_2, \ldots, u_N), \qquad \vec{v} = (v_1, v_2, \ldots, v_N), \\ &\vec{m} = \vec{u} - \vec{u}_{xx}, \qquad \vec{n} = \vec{v} - \vec{v}_{xx}. \end{aligned} \tag{6}$$

Using this notation, Eq. (5) is expressed in the following vector form

$$\begin{cases} \vec{m}_t = (\vec{m}H)_x + \vec{m}H + \frac{1}{(N+1)^2}[\vec{m}(\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x) + (\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T\vec{m}], \\ \vec{n}_t = (\vec{n}H)_x - \vec{n}H - \frac{1}{(N+1)^2}[\vec{n}(\vec{u}-\vec{u}_x)^T(\vec{v}+\vec{v}_x) + (\vec{v}+\vec{v}_x)(\vec{u}-\vec{u}_x)^T\vec{n}], \\ \vec{m} = \vec{u} - \vec{u}_{xx}, \qquad \vec{n} = \vec{v} - \vec{v}_{xx}, \end{cases} \tag{7}$$

where the symbol T denotes the transpose of a vector.

Let us introduce a pair of $(N+1) \times (N+1)$ matrix spectral problems

$$\phi_x = U\phi, \qquad \phi_t = V\phi, \tag{8}$$

with

$$\begin{aligned} &\phi = (\phi_1, \phi_{21}, \ldots, \phi_{2N})^T, \\ &U = \frac{1}{N+1}\begin{pmatrix} -N & \lambda\vec{m} \\ \lambda\vec{n}^T & I_N \end{pmatrix}, \\ &V = \frac{1}{N+1}\begin{pmatrix} -N\lambda^{-2} + \frac{1}{N+1}(\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^T & \lambda^{-1}(\vec{u}-\vec{u}_x) + \lambda\vec{m}H \\ \lambda^{-1}(\vec{v}+\vec{v}_x)^T + \lambda\vec{n}^TH & \lambda^{-2}I_N - \frac{1}{N+1}(\vec{v}+\vec{v}_x)^T(\vec{u}-\vec{u}_x) \end{pmatrix}, \end{aligned} \tag{9}$$

where λ is a spectral parameter, I_N is the $N \times N$ identity matrix, $\vec{u}$, $\vec{v}$, $\vec{m}$ and $\vec{n}$ are the vector potentials shown in (6).

Proposition 1. *(8) provides the Lax pair for the multi-component system (5).*