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Toric Vaisman manifolds

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ABSTRACT

Vaisman manifolds are strongly related to Kähler and Sasaki geometry. In this paper we introduce toric Vaisman structures and show that this relationship still holds in the toric context. It is known that the so-called minimal covering of a Vaisman manifold is the Riemannian cone over a Sasaki manifold. We show that if a complete Vaisman manifold is toric, then the associated Sasaki manifold is also toric. Conversely, a toric complete Sasaki manifold, whose Kähler cone is equipped with an appropriate compatible action, gives rise to a toric Vaisman manifold. In the special case of a strongly regular compact Vaisman manifold, we show that it is toric if and only if the corresponding Kähler quotient is toric.

1. Introduction

Toric geometry has been studied intensively, as manifolds with many symmetries often occur in physics and also represent a large source of examples as testing ground for conjectures. The classical case of compact symplectic toric manifolds has been completely classified by T. Delzant [1], who showed that they are in one-to-one correspondence to the so-called Delzant polytopes, obtained as the image of the momentum map. Afterwards, similar classification results have been given in many different geometrical settings, some of which we briefly mention here. For instance, classification results were obtained by Y. Karshon and E. Lerman [2] for non-compact symplectic toric manifolds and by E. Lerman and S. Tolman [3] for symplectic orbifolds. The case when one additionally considers compatible metrics invariant under the toric action is also well understood: compact toric Kähler manifolds have been investigated by V. Guillemin [4], D. Calderbank, L. David and P. Gauduchon [5], M. Abreu [6], and compact toric Kähler orbifolds in [7]. Other more special structures have been completely classified, such as orthotoric Kähler, by V. Apostolov, D. Calderbank and P. Gauduchon [8] or toric hyperkähler, by R. Bielawski and A. Dancer [9]. The odd-dimensional counterpart, namely the compact contact toric manifolds, are classified by E. Lerman [10], whereas toric Sasaki manifolds were also studied by M. Abreu [11,12]. These were used to produce examples of compact Sasaki–Einstein manifolds, for instance by D. Martelli, J. Sparks and S.-T. Yau [13], A. Futaki, H. Ono and G. Wang [14], C. van Coevering [15].

In the present paper, we consider toric geometry in the context of locally conformally Kähler manifolds. These are defined as complex manifolds admitting a compatible metric, which, on given charts, is conformal to a local Kähler metric. The differentials of the logarithms of the conformal factors glue up to a well-defined closed 1-form, called the Lee form. We are mostly interested in the special class of so-called Vaisman manifolds, defined by the additional property of having parallel Lee form. By analogy to the other geometries, we introduce the notion of toric locally conformally Kähler manifold. More precisely, we require the existence of an effective torus action of dimension half the dimension of the manifold, which

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preserves the holomorphic structure and is twisted Hamiltonian. I. Vaisman [16] introduced twisted Hamiltonian actions and they have been used for instance by S. Haller and T. Rybicki [17] and by R. Gini, L. Ornea and M. Parton [18], where reduction results for locally symplectic, respectively locally conformal Kähler manifolds are given, or more recently by A. Otiman [19].

Vaisman geometry is closely related to both Sasaki and Kähler geometry. In fact, a locally conformally Kähler manifold may be equivalently defined as a manifold whose universal covering is Kähler and on which the fundamental group acts by holomorphic homotheties. For Vaisman manifolds, the universal and the minimal covering are Kähler cones over Sasaki manifolds, as proven in [20,21]. On the other hand, in the special case of strongly regular compact Vaisman manifolds, the quotient by the 2-dimensional distribution spanned by the Lee and anti-Lee vector fields is a Kähler manifold, *cf.* [22].

The purpose of this paper is to make a first step towards the classification of toric Vaisman, or more generally, toric locally conformally Kähler manifolds, by showing that the above mentioned connections between Vaisman and Sasaki, respectively Kähler manifolds, are still true when requiring the toric condition. For the precise statement of these equivalences, we refer to Theorems 4.9, 4.11 and 5.1.

2. Preliminaries

A *locally conformally Kähler manifold* (shortly lcK) is a conformal Hermitian manifold $(M^{2n}, [g], J)$ of complex dimension $n \geq 2$, such that for one (and hence for all) metric g in the conformal class, the corresponding fundamental 2-form $\omega := g(\cdot, J\cdot)$ satisfies: $\mathrm{d}\omega = \theta \wedge \omega$, with θ a closed 1-form, called the *Lee form* of the Hermitian structure (g, J). Equivalently, there exists an atlas on M, such that the restriction of g to any chart is conformal to a Kähler metric. In fact, the differential of the logarithms of the conformal factors are, up to a constant, equal to the Lee form. It turns out to be convenient to denote also by (M, g, J, θ) an lcK manifold, when fixing one metric g in the conformal class. By ∇ we denote the Levi-Civita connection of g.

We denote by $\theta^\sharp$ the vector field dual to θ with respect to the metric g, the so-called *Lee vector field* of the lcK structure, and by $J\theta^\sharp$ the *anti-Lee vector field*.

Remark 2.1. On an lcK manifold (M, g, J, θ), the following formula for the covariant derivative of J holds:

$$2\nabla_X J = X \wedge J\theta^\sharp + JX \wedge \theta^\sharp, \quad \forall X \in \mathfrak{X}(M),$$

or, more explicitly, applied to any vector field $Y \in \mathfrak{X}(M)$:

$$2(\nabla_X J)(Y) = \theta(JY)X - \theta(Y)JX + g(JX, Y)\theta^\sharp + g(X, Y)J\theta^\sharp. \tag{1}$$

In particular, it follows that $\nabla_{\theta^\sharp} J = 0$ and $\nabla_{J\theta^\sharp} J = 0$.

Remark 2.2. On an lcK manifold (M^{2n}, g, J, θ), a vector field X preserving the fundamental 2-form ω, also preserves the Lee form, *i.e.* $\mathcal{L}_X\omega = 0$ implies $\mathcal{L}_X\theta = 0$, as follows. As the differential and the Lie derivative with respect to a vector field commute to each other, *e.g.* by the Cartan formula, we obtain:

$$0 = \mathrm{d}(\mathcal{L}_X\omega) = \mathcal{L}_X(\mathrm{d}\omega) = \mathcal{L}_X(\theta \wedge \omega) = \mathcal{L}_X\theta \wedge \omega + \theta \wedge \mathcal{L}_X\omega = \mathcal{L}_X\theta \wedge \omega.$$

Since the map form $\Omega^1(M)$ to $\Omega^3(M)$ given by wedging with ω is injective, for complex dimension $n \geq 2$, it follows that $\mathcal{L}_X\theta = 0$.

We now recall the definition of Vaisman manifolds, which were first introduced and studied by I. Vaisman [20,22]:

Definition 2.3. A *Vaisman manifold* is an lcK manifold $(M, [g], J)$ admitting a metric in the conformal class, such that its Lee form is non-zero and parallel with respect to the Levi-Civita connection of the metric.

Note that on a compact lcK manifold, a metric with parallel Lee form θ, if it exists, is unique up to homothety in its conformal class and coincides with the so-called *Gauduchon metric*, *i.e.* the metric with co-closed Lee form: $\delta\theta = 0$. In this paper, we scale any Vaisman metric g such that the norm of its Lee vector field $\theta^\sharp$, which is constant since θ is parallel, equals 1.

Definition 2.4. The automorphism group of a Vaisman manifold (M, g, J, θ) is denoted by a slight abuse of notation $\mathrm{Aut}(M) := \mathrm{Aut}(M, g, J, \theta)$ and is defined as the group of conformal biholomorphisms:

$$\mathrm{Aut}(M) = \{F \in \mathrm{Diff}(M) \mid F^*J = J, [F^*g] = [g]\}.$$

We emphasize here that we define the group of automorphisms like for lcK manifolds, namely we do not ask for an automorphism of a Vaisman manifold to be an isometry of the Vaisman metric, but only to preserve its conformal class. Hence, the Lie algebra of $\mathrm{Aut}(M)$ is:

$$\mathfrak{aut}(M) = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X J = 0, \mathcal{L}_X g = fg, \text{ for some } f \in \mathcal{C}^\infty(M)\}. \tag{2}$$

We denote by $\mathfrak{isom}(M)$ and $\mathfrak{hol}(M)$ the Lie algebras of Killing vector fields with respect to the Vaisman metric g, respectively of holomorphic vector fields on (M, J).