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Limit equation for vacuum Einstein constraints with a translational Killing vector field in the compact hyperbolic case

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ABSTRACT

We construct solutions to the constraint equations in general relativity using the limit equation criterion introduced in Dahl et al. (2012). We focus on solutions over compact 3-manifolds admitting a $\mathbb{S}^1$-symmetry group. When the quotient manifold has genus greater than 2, we obtain strong far from CMC results.

1. Introduction

General relativity describes the universe as a $(3+1)$-dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric $\mathbf{g}$. The Einstein equations describe how non-gravitational fields influence the curvature of $\mathbf{g}$:

$$\mathbf{Ric}_{\mu\nu} - \frac{\mathbf{Scal}}{2}\mathbf{g}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu},$$

where **Ric** and **Scal** are respectively the Ricci tensor and the scalar curvature of the metric $\mathbf{g}$ and $\mathbf{T}_{\mu\nu}$ is the sum of the energy–momentum tensors of all the non-gravitational fields.

Einstein equations can be formulated as a Cauchy problem with initial data given by a set $(M, \widehat{g}, \widehat{K})$, where M is a 3-dimensional manifold, $\widehat{g}$ is a Riemannian metric on M and $\widehat{K}$ is a symmetric 2-tensor on M. $\widehat{g}$ and $\widehat{K}$ correspond to the first and second fundamental forms of M seen as an embedded space-like hypersurface in the universe $(\mathcal{M}, \mathbf{g})$ solving the Einstein equations.

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It turns out that the Einstein equations imply compatibility conditions on $\widehat{g}$ and $\widehat{K}$ known as the constraint equations:

$$\begin{cases}\mathrm{Scal}_{\widehat{g}} + (\mathrm{tr}_{\widehat{g}}\widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}} = 2\rho, & \text{(a)}\\ \mathrm{div}_{\widehat{g}}\widehat{K} - d(\mathrm{tr}_{\widehat{g}}\widehat{K}) = j, & \text{(b)}\end{cases} \tag{1.1}$$

where, denoting by N the unit future-pointing normal to M in $\mathcal{M}$, one has

$$\rho = 8\pi \mathbf{T}_{\mu\nu}N^\mu N^\nu, \qquad j_i = 8\pi \mathbf{T}_{i\mu}N^\mu.$$

We assume here that μ and ν go from 0 to 3 and denote spacetime coordinates while Latin indices go from 1 to 3 and correspond to coordinates on M.

In this article, to keep things simple, we will consider no field but the gravitational one (vacuum case). As a consequence, we impose $\mathbf{T} \equiv 0$. We will also assume that the spacetime possesses a $\mathbb{S}^1$-symmetry generated by a spacelike Killing vector field. This allows for a reduction of the $(3+1)$-dimensional study of the Einstein equations to a $(2+1)$-dimensional problem. This symmetry assumption has been introduced and studied by Choquet-Bruhat and Moncrief in [1] (see also [2]) in the case of a spacetime of the form $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a compact 2-dimensional manifold of genus $G \geq 2$, $\mathbb{S}^1$ corresponds to the orbit of the $\mathbb{S}^1$-action and $\mathbb{R}$ is the time axis. They proved the existence of global solutions corresponding to perturbations of a particular expanding spacetime. In [1], they use solutions of the constraint equations with constant mean curvature (CMC, i.e. constant $\mathrm{tr}_{\widehat{g}}\widehat{K}$) on the spacelike hypersurface $\Sigma \times \mathbb{S}^1 \times \{0\}$ as initial data. The construction of such solutions is fairly direct. In this article we shall generalize their construction to more general initial data allowing for non-constant mean curvature.

The method which is generally used to construct initial data for the Einstein equations is the conformal method which consists in decomposing the metric $\widehat{g}$ and the second fundamental form $\widehat{K}$ into given data and unknowns that have to be adjusted so that $\widehat{g}$ and $\widehat{K}$ solve the constraint equations, see Section 2. The equations for the unknowns, namely a positive function playing the role of a conformal factor and a 1-form, are usually called the conformal constraint equations. Extended discussion of the conformal method can be found in a series of very nice articles by D. Maxwell [3–6].

These equations have been extensively studied in the case of constant mean curvature (CMC) since the system greatly simplifies in this case. We refer the reader to the excellent review article [7] for an overview of known results in this particular case. The non-CMC case remained open for a couple of decades. Only the case of nearly constant mean curvature was studied. We refer for example to the pioneer work of Isenberg and Moncrief [8]. Two major breakthroughs were obtained in [9,10] and [11] concerning the far from CMC case. A comparison of these methods is given in [12].

In this article, we follow the method described in [11]. Namely, we give the following criterion: if a certain limit equation admits no non-zero solution, the conformal constraint equations admit at least one solution. The other method [9,10] would require that Σ is $\mathbb{S}^2$ so that it carries a metric with positive scalar curvature and has no conformal Killing vector field, which is impossible.

This approach has been generalized to the asymptotically hyperbolic case in [13] and to the asymptotically cylindrical case in [14]. The asymptotically Euclidean case [15] and the case of compact manifolds with boundary [16] are currently work in progress since new ideas have to be found to get the criterion.

The outline of the paper is as follows. In Section 2, we show how the Einstein equations reduce to a $(2+1)$-dimensional problem in the case of a $\mathbb{S}^1$-symmetry and exhibit the analog of the conformal constraint equations in this case. We also state Theorem 2.1 which is the main result of this article and Corollary 2.3 which gives an example of application of Theorem 2.1. Section 3 is devoted to the proof of Theorem 2.1. Finally, Section 4 contains the proof of Corollary 2.3.

2. Preliminaries

2.1. Reduction of the Einstein equations

Before discussing the constraint equations, we briefly recall the form of the Einstein equations in the presence of a spacelike translational Killing vector field. We follow here the exposition in [2, Section XVI.3].

We recall that we want to write the Einstein equations on the manifold $\mathcal{M} = \Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a Riemannian surface and $\mathbb{R}$ denotes the time direction, for some metric $\mathbf{g}$ which is invariant under translation along the $\mathbb{S}^1$-direction. We let x^3 denote the coordinate along the $\mathbb{S}^1$- direction (seen as $\mathbb{R}/\mathbb{Z}$), choose local coordinates x^1, x^2 on Σ and denote by x^0 the time coordinate.

A metric $\mathbf{g}$ on $\mathcal{M}$ admitting ∂_3 as a Killing vector field has the form

$$\mathbf{g} = \widetilde{g} + e^{2\gamma}\left(dx^3 + A\right)^2,$$

where $\widetilde{g}$ is a Lorentzian metric on $\Sigma \times \mathbb{R}$, A is a 1-form on $\Sigma \times \mathbb{R}$ and γ is a function on $\Sigma \times \mathbb{R}$. Since ∂_3 is a Killing vector field, $\widetilde{g}$, A and γ do not depend on x^3. We set $F = dA$ the field strength of A. The Ricci tensor $\mathbf{Ric}$ of $\mathbf{g}$ can be computed in