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Parallel transport on principal bundles over stacks

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ABSTRACT

In this paper we introduce a notion of parallel transport for principal bundles with connections over differentiable stacks. We show that principal bundles with connections over stacks can be recovered from their parallel transport thereby extending the results of Barrett, Caetano and Picken, and Schreiber and Waldorf from manifolds to stacks.

In the process of proving our main result we simplify Schreiber and Waldorf's original definition of a transport functor for principal bundles with connections over manifolds and provide a more direct proof of the correspondence between principal bundles with connections and transport functors.

1. Introduction

Let G be a Lie group and M a C^∞ manifold. Recall that a choice of a connection 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ on a principal G-bundle P over the manifold M and a choice of a base point $x \in M$ gives rise to the holonomy map

$$\Omega(M, x) \to \mathrm{Aut}(\text{fiber of } P \text{ at } x) \simeq G,$$

where $\Omega(M, x)$ is the set of smooth loops at x in M. For a connected manifold M holonomy map uniquely determines the connection A and, in fact, the bundle P itself [1]. If two loops in $\Omega(M, x)$ differ by a homotopy that sweeps no area, a so-called "thin homotopy", then their holonomies are the same. Therefore the holonomy map descends to a well defined map on the quotient

$$\mathcal{H} : \Omega(M, x)/_\sim \to G,$$

where $\sim$ means "identify thinly homotopic loops". The quotient $\pi_1^{\mathrm{thin}}(M, x) := \Omega(M, x)/_\sim$ is a group and $\mathcal{H}$ is a homomorphism. Moreover $\pi_1^{\mathrm{thin}}(M, x)$ has a smooth structure – it is a diffeological group (see Appendix A and Remark 2.13) – and $\mathcal{H}$ is smooth. Barrett [2], motivated by questions coming from general relativity and Yang–Mills theory, proved that a homomorphism

$$T : \pi_1^{\mathrm{thin}}(M, x) \to G$$

is defined by parallel transport on some principal G-bundle with connection if and only if T is smooth. More precisely, he proved that assigning parallel transport homomorphisms to a principal bundle with a connection induces a bijection of sets:

$$\begin{aligned}&(\text{principal bundles with connections over } M)/\text{isomorphisms}\\ &\quad\leftrightarrow \quad (\text{smooth homomorphisms } \pi_1^{\mathrm{thin}}(M, x) \to G)/\text{conjugation}.\end{aligned}$$

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Barrett's proofs were simplified by Caetano and Picken [3]. Wood [4] reformulated Barrett's theorem in terms of the groupoids of paths in M; this obviates the need to choose a base point. Schreiber and Waldorf [5] categorified Wood's version of Barrett's theorem. They showed that assigning holonomy to a bundle defines an equivalence of categories

$$\mathsf{hol}_M : B^\nabla G(M) \to \mathsf{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G\text{-tor}).$$

Here, and in the rest of the paper, $B^\nabla G(M)$ denotes the category of principal G-bundles with connections over a manifold M, $\Pi^{\text{thin}}(M)$ is the thin fundamental groupoid of M (see Definition/Proposition 2.9), G-tor is the category of G-torsors (Definition 3.2) and $\mathsf{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G\text{-tor})$ denotes a category of functors that are smooth in an appropriate sense (see Definition 3.5). Schreiber and Waldorf's definition of $\mathsf{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G\text{-tor})$ is fairly involved and the proof that hol_M is an equivalence of categories is indirect. Nor is it clear if the equivalence hol_M is natural in the manifold M.

In this paper we propose a simple definition of what it means for a functor $T : \Pi^{\text{thin}}(M) \to G$-tor to be smooth. We refer to such smooth functors as *transport functors*. We shorten our notation by setting

$$\mathsf{Trans}_G(M) := \mathsf{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G\text{-tor});$$

$\mathsf{Trans}_G(M)$ is a category whose objects are parallel transport functors and morphisms are natural transformations (see Definition 3.16). We provide a sanity check by showing that the parallel transport functor $\mathsf{hol}_M(P, A)$ defined by a connection A on a principal G-bundle $P \to M$ is smooth in the sense of this paper. We then prove that for a manifold M the functor hol_M is an equivalence of categories (Theorem 4.1). This part of the paper does not require any knowledge of stacks.

In the second part of the paper we assume that the reader is familiar with stacks over the site Man of manifolds. The standard references are Behrend and Xu [6], Heinloth [7] and Metzler [8].

We first prove that the assignment $M \mapsto \mathsf{Trans}_G(M)$ extends to a contravariant functor $\mathsf{Trans}_G : \mathsf{Man}^{op} \to \mathsf{Groupoid}$ (Lemma 5.1). By a Grothendieck construction, the presheaf Trans_G defines a category fibered in groupoids (CFG) $\underline{\mathsf{Trans}}_G \to \mathsf{Man}$. The collection of functors

$$\{\mathsf{hol}_M : B^\nabla G(M) \to \mathsf{Trans}_G(M)\}_{M \in \mathsf{Man}}$$

extends to a morphism of CFGs $\mathsf{hol} : B^\nabla G \to \underline{\mathsf{Trans}}_G$ (Lemma 5.3). Since each functor hol_M is an equivalence of categories, the functor hol is an equivalence (Theorem 5.4). Consequently, since $B^\nabla G$ is a stack, so is $\underline{\mathsf{Trans}}_G$ (Corollary 5.5). Together the two results imply one of the main results of the paper:

$$\mathsf{hol} : B^\nabla G \to \underline{\mathsf{Trans}}_G$$

is an isomorphism of stacks. In Section 6 we work out some consequence of Theorem 5.4 for principal bundles with connections over stacks. We start by recalling a definition of a principal G-bundle over a stack $\mathcal{X}$: it is a functor $P : \mathcal{X} \to BG$, where BG denotes the stack of principal G bundles. By analogy we introduce the notion of a principal bundle with connection and of a transport functor over a stack $\mathcal{X}$. Note that we do not assume that $\mathcal{X}$ necessarily has an atlas. As an immediate consequence of Theorem 5.4, we obtain that for each stack $\mathcal{X}$ the functor hol induces an equivalence of categories between the categories of principal bundles with connections over $\mathcal{X}$ and transport functors over $\mathcal{X}$ (Theorem 6.4). We then recall that for a CFG $\mathcal{X} \to \mathsf{Man}$ and a Lie groupoid Γ, there is the category $\mathcal{X}(\Gamma)$ of cocycles with values in $\mathcal{X}$. We discuss the fact that the cocycle category $\mathcal{X}(\Gamma)$ is equivalent to the functor category $[[\Gamma_0/\Gamma_1], \mathcal{X}]$ (Proposition 6.6). Here and elsewhere in the paper $[\Gamma_0/\Gamma_1]$ denotes the stack quotient of the Lie groupoid Γ. We end Part 2 of the paper by reformulating Theorem 6.4 in terms of the cocycle categories: for any Lie groupoid Γ the isomorphism of stacks $\mathsf{hol} : B^\nabla G \to \underline{\mathsf{Trans}}_G$ induces an equivalence $\mathsf{hol}_\Gamma : B^\nabla G(\Gamma) \to \underline{\mathsf{Trans}}_G(\Gamma)$ of the cocycle categories (Theorem 6.7).

The paper has two appendices. In Appendix A we review the definition of a diffeological space both from a traditional point of view and as a concrete sheaf of sets. We prove the folklore result that the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ of a manifold M is a diffeological groupoid. We also prove two technical results that are needed elsewhere in the paper. We show that the target map $\mathfrak{t}$ of the thin fundamental groupoid has local sections (Lemma A.26). We prove that the assignment $M \mapsto \Pi^{\text{thin}}(M)$ extends to a functor $\Pi^{\text{thin}} : \mathsf{Man} \to \mathsf{DiffGpd}$ from the category of manifolds to the category $\mathsf{DiffGpd}$ of diffeological groupoids. In Appendix B we prove that for any Lie groupoid Γ an equivalence of CFGs $F : \mathcal{X} \to \mathcal{Y}$ induces an equivalence $F_\Gamma : \mathcal{X}(\Gamma) \to \mathcal{Y}(\Gamma)$ of the corresponding cocycle categories (Proposition 6.8).

Part 1. Parallel transport for bundles over manifolds

2. Thin homotopy and the thin fundamental groupoid

In this section, following Schreiber and Waldorf, we define the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ of a manifold M. Nothing in this section is new. Our purpose for presenting this material is to keep the paper self-contained and to fix notation. To start we recall the notion of a path with sitting instances of Caetano and Picken [3].

Definition 2.1 (*A Path with Sitting Instances*). Let $[a, b] \subset \mathbb{R}$ be a closed interval and M a manifold. A smooth map $\gamma : [a, b] \to M$ is a path with sitting instances if γ is constant on neighborhoods of a and b.