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High values of disorder-generated multifractals and logarithmically correlated processes

Yan V. Fyodorov [a], Olivier Giraud [b,*]

[a] School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom
[b] LPTMS, CNRS and Université Paris-Sud, UMR 8626, Bât. 100, 91405 Orsay, France

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ABSTRACT

In the introductory section of the article we give a brief account of recent insights into statistics of high and extreme values of disorder-generated multifractals following a recent work by the first author with P. Le Doussal and A. Rosso (FLR) employing a close relation between multifractality and logarithmically correlated random fields. We then substantiate some aspects of the FLR approach analytically for multifractal eigenvectors in the Ruijsenaars–Schneider ensemble (RSE) of random matrices introduced by E. Bogomolny and the second author by providing an *ab initio* calculation that reveals hidden logarithmic correlations at the background of the disorder-generated multifractality. In the rest we investigate numerically a few representative models of that class, including the study of the highest component of multifractal eigenvectors in the Ruijsenaars–Schneider ensemble.

1. Introduction

1.1. General setting

Multifractal patterns are patterns of intensities which are characterized by a high variability over a wide range of space or time scales, and by huge fluctuations which can be visually detected. They have been observed and investigated in many areas of science, from physics, chemistry, geophysics, oceanology [1,2] to climate studies [3] or mathematical finance [4,5]. The multifractal approach has also proved relevant in fields such as growth processes [8], turbulence [6,7], and the theory of quantum disordered systems [9] (See Fig. 1).

In a d-dimensional lattice of linear size L and lattice spacing a, thus containing $M=(L/a)^d \gg 1$ lattice sites, multifractal patterns with intensities $h_i > 0$ at different sites $i=1,\dots,M$ are characterized by attributing a different scaling $h_i \sim M^{x_i}$ to each intensity, with exponents x_i forming a dense set. One of the most natural characteristics of a multifractal is the function $\mathcal{N}_M(x)$ counting the number of points in the pattern with exponents exceeding the value x. Introducing the density of exponents $\rho_M(x)$, so that $\mathcal{N}_M(x)=\int_x^\infty \rho_M(y)\,dy$, multifractality is equivalent to the statement that such a density behaves for $M \gg 1$ as

$$\rho_M(x)=\sum_{i=1}^{M}\delta\left(\frac{\ln h_i}{\ln M}-x\right)\sim c_M(x)\sqrt{\ln M}\,M^{f(x)},\quad M\gg 1, \quad (1)$$

where $f(x)$, the singularity spectrum, is a function of x, and $c_M(x)$ is of order unity. This is frequently referred to as the *multifractal Ansatz*. The characteristic feature of multifractal patterns in systems with disorder, like Anderson localisation transition and related phenomena, is the existence of essential sample-to-sample fluctuations of the prefactor $c_M(x)$ in different realizations of the disorder, as well as fluctuations in the number and height of extreme peaks of the pattern. Those fluctuations will be the subject of our interest. At the same time multifractality can be studied via the singularity spectrum $f(x)$ [10,11]. It is typically a self-averaging convex function like the one shown in Fig. 2. Some general insight into statistical

* Corresponding author.

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properties of disordered multifractals have been obtained in [12] and the content of that work is concisely summarized below.

As is well-known [13,14] disorder-generated multifractal patterns of intensities $h(\mathbf{r})$ are typically *self-similar*, i.e. characterized by the power-law correlation of intensities

$$\mathbb{E}\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\} \propto \left(\frac{L}{a}\right)^{y_{q,s}} \left(\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}\right)^{-z_{q,s}}, \quad q,s \geqslant 0, \quad a \ll |\mathbf{r}_1-\mathbf{r}_2| \ll L, \tag{2}$$

and *spatially homogeneous*

$$\mathbb{E}\{h^q(\mathbf{r})\} = \mathbb{E}\left\{\frac{1}{M}\sum_{\mathbf{r}} h^q(\mathbf{r})\right\} \propto \left(\frac{L}{a}\right)^{d(\zeta_q-1)}, \tag{3}$$

where here and henceforth $\mathbb{E}\{A\}$ stands for the expected value (the mean) of the random variable A. The lattice model describes a situation where the relevant scales are L and a, therefore it is natural to assume that intensities do not vary much over the scale a and that they are uncorrelated at scale L. This can be expressed as

$$\mathbb{E}\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\} \sim \mathbb{E}\{h^{q+s}(\mathbf{r}_1)\} \quad |\mathbf{r}_1-\mathbf{r}_2| \sim a, \tag{4}$$

$$\mathbb{E}\{h^q(\mathbf{r}_1)h^s(\mathbf{r}_2)\} \sim \mathbb{E}\{h^q(\mathbf{r}_1)\}\mathbb{E}\{h^s(\mathbf{r}_2)\} \quad |\mathbf{r}_1-\mathbf{r}_2| \sim L. \tag{5}$$

If we make the assumption that Eq. (2) holds over the whole range $|\mathbf{r}_1-\mathbf{r}_2| \sim a$ to $|\mathbf{r}_1-\mathbf{r}_2| \sim L$, we directly get from (4) and (5) the relations between exponents

$$y_{q,s} = d(\zeta_{q+s}-1), \quad z_{q,s} = d(\zeta_{q+s}-\zeta_q-\zeta_s+1), \tag{6}$$

so that the set of exponents ζ_q is the only one needed to characterize the spatial organization of such a multifractal pattern [13,14].

It proves to be instructive to shift the focus from the multifractal field $h(\mathbf{r})$ to its logarithm $V(\mathbf{r}) = \ln h(\mathbf{r}) - \mathbb{E}\{\ln h(\mathbf{r})\}$. Correlations of the field $V(\mathbf{r})$ can be obtained by deriving $\langle h^q h^s\rangle - \langle h^q\rangle\langle h^s\rangle$, given by Eqs. (2) and (3), with respect to q and s, using the identity $\frac{d}{ds}h^s|_{s=0} = \ln h$. Taking into account the relations (6) and the fact that $\zeta_0 = 1$ one arrives at the relation [15]

$$\mathbb{E}\{V(\mathbf{r}_1)V(\mathbf{r}_2)\} = -d\,\zeta_0''\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{L}, \tag{7}$$

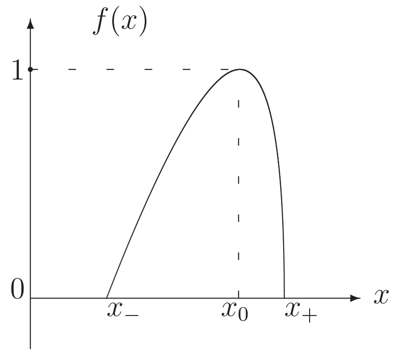

Fig. 2. Shape of a typical singularity spectrum.

where ζ_0'' is the second derivative of ζ_q taken at $q = 0$. We thus conclude that provided the conditions (2) and (3) of self-similarity and spatial homogeneity detailed above are fulfilled, the logarithm of a disorder-generated multifractal intensity must be necessarily a *log-correlated* random field [15]. Note that the nature of the higher cumulants is not fixed by this construction, and in particular there is no particular reason to expect Gaussianity of the field $V(\mathbf{r})$ on general grounds. Moreover, had the field been Gaussian the only possible shape of the singularity spectrum $f(x)$ would be a simple parabola. In practice, non-parabolic shapes are abundant in disordered multifractals [9], although shapes extremely close to perfect parabolas also occur, most notably in the Integer Quantum Hall context [16].

The shift of attention from the multifractal field to its logarithm is of conceptual and practical utility as extremes of random fields and processes with logarithmic correlations attracted recently a lot of attention in physics [17–19], probability [20,21] and related areas. The most studied object is the 2D Gaussian free field (GFF) which is now believed to be as fundamental and rich as Brownian motion, and naturally emerges in studies ranging from quantum gravity and turbulence to financial mathematics. One of the most powerful rigorous frameworks for analyzing such fields and related processes relies upon the theory of "multiplicative chaos" [22]. Another important source of

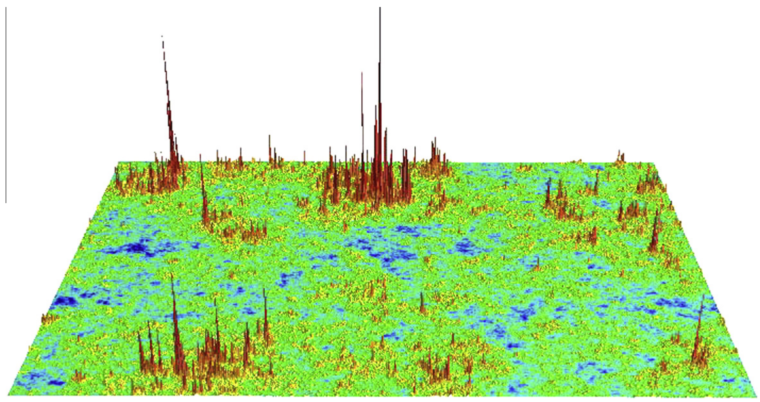

Fig. 1. Intensity of a multifractal wavefunction at the point of Integer Quantum Hall Effect. Courtesy of F. Evers, A. Mirlin and A. Mildenberger.