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Sampling local properties of attractors via Extreme Value Theory

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ABSTRACT

We provide formulas to compute the coefficients entering the affine scaling needed to get a non-degenerate function for the asymptotic distribution of the maxima of some kind of observable computed along the orbit of a randomly perturbed dynamical system. This will give information on the local geometrical properties of the stationary measure. We will consider systems perturbed with additive noise and with observational noise. Moreover we will apply our techniques to chaotic systems and to contractive systems, showing that both share the same qualitative behavior when perturbed.

1. Introduction

A general problem in dynamical systems theory is to give a quantitative characterization of the limiting invariant sets like attractors or repellers, whose properties are essential to understand the behavior of complex systems. In the last years, the results of the Extreme Value Theory (EVT) have brought new techniques that allow to quantify the geometrical and dynamical properties of a certain class of systems. In the case of absolutely continuous invariant measures (acim), precise analytical results can be obtained in terms of classical extreme value laws (EVLs) and depend on the fulfillment of general mixing conditions and on the observables considered [1–4]. In fact, those observable are designed in such a way that *extreme events* are equivalent to detect the recurrence of an orbit in a neighborhood of a given point in the phase space. A collection of such events, under appropriate renormalization, is distributed according to one of the three classical EVLs, namely the Gumbel, the Frechet and the Weibull distributions. The values of the normalizing constants are linked to the local behavior of the measure and, provided the dynamics is chaotic and the measure is absolutely continuous, they depend only on the number of extremes extracted and on the phase space dimension. Several difficulties arise whenever singular invariant measures are considered. In [5,6] this problem was addressed almost numerically and a few analytic results have been exhibited in [7,4,8,9].

Let us now explain in detail where the just indicated problem is and how we could deal with it by introducing random perturbations: this will constitute the first main contribution of this paper. At this regard we need to come back to basics and introduce the theory. Let us therefore suppose that $(Y_n)_{n\in\mathbb{N}}$ is a sequence of real-valued random variables defined on the probability space $(\Psi, \mathbb{P})$. We will be interested in the distribution of the maximum $M_n := \max\{Y_0, Y_1, \dots, Y_{n-1}\}$ when $n \to \infty$. It is well known that the limiting distribution is degenerate unless one proceed to a suitable re-scaling of the levels of exceedances.

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The precise formulation is the following: we have an extreme value law for $(M_n)_{n\in\mathbb{N}}$ if there is a non-degenerate distribution function $H:\mathbb{R}\to[0,1]$ with $H(0)=0$ and, for every $\tau>0$, there exists a sequence of levels $(u_n(\tau))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} n\mathbb{P}(Y_0>u_n)\to\tau, \tag{1.1}$$

and for which the following holds:

$$\lim_{n\to\infty}\mathbb{P}(M_n\leqslant u_n)\to 1-H(\tau).$$

The motivation for using a normalizing sequence $(u_n)_{n\in\mathbb{N}}$ satisfying (1.1) comes from the case when $(Y_n)_{n\in\mathbb{N}}$ are independent and identically distributed. In this i.i.d. setting, it is clear that $\mathbb{P}(M_n\leqslant u)=(F(u))^n$, being $F(u)$ the cumulative distribution function for the variable u. Hence, condition (1.1) implies that

$$\mathbb{P}(M_n\leqslant u_n)=(1-\mathbb{P}(Y_0>u_n))^n\sim\left(1-\frac{\tau}{n}\right)^n\to e^{-\tau},$$

as $n\to\infty$. Note that in this case $H(\tau)=1-e^{-\tau}$ is the standard exponential distribution function. Let us now choose the sequence $u_n=u_n(y)$ as the one parameter family $u_n=y/a_n+b_n$, where $y\in\mathbb{R}$ and $a_n>0$, for all $n\in\mathbb{N}$. Whenever the variables Y_i are i.i.d. and for some constants $a_n>0, b_n\in\mathbb{R}$, we have $\mathbb{P}(a_n(M_n-b_n)\leqslant y)\to G(y)$, where the convergence occurs at continuity points of G, and G is non-degenerate, then G_n will converge to one of the three EVLs: Gumbel, Fréchet or Weibull. The law obtained depends on the common distribution of the random variables, F.

When $Y_0, Y_1, Y_2,\ldots$ are not independent, the standard exponential law still applies under some conditions on the dependence structure. These conditions will be stated in detail later and they are usually designated by D_2 and D'; when they hold for $(Y_n)_{n\in\mathbb{N}}$ then there exists an extreme value law for M_n and $H(\tau)=1-e^{-\tau}$, see Theorem 1 in [10]. We want to stress that these two conditions alone do not imply the existence of an extreme value law; they require, even to be checked, that the limit (1.1) holds. It turns out that for the kind of observables we are going to introduce, and which are related to the local properties of the invariant measure, the limit (1.1) is difficult to prove when the invariant measure is not absolutely continuous, since one needs the exact asymptotic behavior of that measure on small balls. Instead it turns out that whenever the system is randomly perturbed, the limit (1.1) is more accessible and in particular it will be given by a closed formula in terms of the strength of the noise, see Propositions 1 and 2 below. Moreover that formula could be used in a reversed way (in the following we call this procedure *inverting the technique*): since the sequence $(u_n)_{n\in\mathbb{N}}$ is now uniquely determined for any n, a numerical sampling for $(u_n)_{n\in\mathbb{N}}$ which provides convergence to the extreme value law, will bring information on the local geometrical properties of the stationary measure: this approach was successfully used, for instance, in [11,6].

We already showed in a preceding article [18] that random perturbations of regular systems, in particular rotations, induce the appearance of extreme value laws since the perturbed systems acquires a chaotic behavior. We pursue, and this is the second main issue of this paper, the same objective here by considering two different kinds of stochastic perturbations of (piecewise) contracting maps. The first will be given by additive noise and in this case our analysis will be mostly numerical. The second one will be a sort of (rare) random contamination of a deterministic orbit, and in this case we will announce and state complete analytic results for the determination of the limit (1.1) first, and for the successive checking of the conditions D_2 and D'.

2. Random dynamical systems

In this section we will introduce the two ways of perturbing a given dynamical system, the random transformations and the observational noise.

2.1. Random transformations

Let us consider a sequence of i.i.d. random variables $(W_k)_{k\in\mathbb{N}}$ with values $(\omega_k)_{k\in\mathbb{N}}$ in a space Ω_ε and with common probability distribution θ_ε. Let $X\subset\mathbb{R}^d$ be a compact set equipped with the Lebesgue measure m defined on the Borel σ-algebra, and $(f_\omega)_{\omega\in\Omega_\varepsilon}$ a family of measurable transformations such that $f_\omega: X\to X$ for all $\omega\in\Omega_\varepsilon$.[1] Given a point $x\in X$ and a realization $\underline{\omega}=(\omega_1,\omega_2,\ldots)\in\Omega_\varepsilon^{\mathbb{N}}$ of the stochastic process $(W_k)_{k\in\mathbb{N}}$, we define the random orbit of x as the sequence $(f^n_{\underline{\omega}}(x))_{n\in\mathbb{N}}$, where

$$f^0_{\underline{\omega}}(x)=x \quad\text{and}\quad f^n_{\underline{\omega}}(x)=f_{\omega_n}\circ f_{\omega_{n-1}}\circ\cdots\circ f_{\omega_1}(x)\quad\forall n\geqslant 1.$$

The transformations f_ω will be considered as stochastic perturbations of a deterministic map f, in the sense that they will be taken in a suitable neighborhood of f whose *size* will be determined by the value of ε, see below. We could therefore define a Markov process on X with transition function

$$L_\varepsilon(x,A)=\int_{\Omega_\varepsilon}\mathbf{1}_A(f_\omega(x))d\theta_\varepsilon(\omega), \tag{2.1}$$

where $A\in X$ is a measurable set, $x\in X$ and $\mathbf{1}_A$ is the indicator function of the set A. A probability measure μ_ε is called *stationary* if for any measurable set A we have:

$$\mu_\varepsilon(A)=\int_X L_\varepsilon(x,A)d\mu_\varepsilon(x).$$

We call it an absolutely continuous stationary measure (*acsm*), if it has a density with respect to the Lebesgue measure.

Given a map $f:X\to X$, we will consider two kind of random perturbations. The first one is the additive noise, which corresponds to the family $(f_\omega)_{\omega\in\Omega_\varepsilon}$ of random transformations defined by

$$f_\omega(x)=f(x)+\omega\quad\forall x\in X.$$

[1] In the following when we will refer to a dynamical system (X,f,μ) we will mean that f is defined on X and preserves the Borel probability measure μ; if we will write (X,f), this will simply correspond to the action of f on X.