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Records in the classical and quantum standard map

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ABSTRACT

Record statistics is the study of how new highs or lows are created and sustained in any dynamical process. The study of the highest or lowest records constitute the study of extreme values. This paper represents an exploration of record statistics for certain aspects of the classical and quantum standard map. For instance the momentum square or energy records is shown to behave like that of records in random walks when the classical standard map is in a regime of hard chaos. However different power laws is observed for the mixed phase space regimes. The presence of accelerator modes are well-known to create anomalous diffusion and we notice here that the record statistics is very sensitive to their presence. We also discuss records in random vectors and use it to analyze the *quantum* standard map via records in their eigenfunction intensities, reviewing some recent results along the way.

1. Introduction

Breaking a record, or setting a new one has been a human passion, perhaps one may say weakness, for a while now. Evidence how one yearns for a record to be set by a particular sportsperson or the euphoria (depression) that sets in when the stock market fluctuations reach a maximum (minimum) never seen before. Of course our view of nature also abounds in such characterizations: the highest mountains, the deepest oceans, the smallest temperatures reached and so on. While these may seem to be questions about extremes, records are a record of extremes, the times they last and the number of new ones created. For engineers designing a dam, one important piece of information is when and by how much the water level of the river has exceeded all its previous values; similar questions are naturally asked about rainfall when planning for agricultural policies is taken up. Practical considerations such as these have been the early applications in a study of events which exceed themselves in some aspect and are often called as "record statistics", akin to the study of extreme statistics in many ways. An accessible introduction to record statistics is found in [36].

Questions about how the records increase with time, or the number of records set, are of natural interest in all these complex, sometimes social contexts, have therefore been studied for example, [12,39,31]. A mathematical theory of records for independent identically distributed (*i.i.d.*) random variables has been developed since the pioneering work of Rényi [32] for example developed in [13,1]. The applications in Physics started somewhat later, but have by now found use in various problems related to random walks, spin-glasses, type II superconductors and quantum chaos. In this work, our emphasis is on the record statistics in deterministic dynamical systems, both classical and quantum. In particular we choose the most studied low-dimensional paradigm of Hamiltonian chaos, namely the standard map, or known variously as the kicked-rotor, Chirikov–Taylor map etc. Its quantization has also been extensively studied, as well as realized experimentally in cold atom set ups in [28].

Now to define records more precisely, given that $\{X_t, t = 1, \ldots, N\}$ is a finite time series, the first element,

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$R(1)$, of the corresponding records series is X_1 itself and at subsequent times t it will be $R(t) = \max(X_t, R(t-1))$. As X_t is a random variable, so is $R(t)$ and properties of this random variable is of interest. Thus $R(t)$ which consists of elements of X_t in a non-decreasing order are the upper record sequence. If the minimum value is taken instead of the maximum the record sequence is called the lower record sequence. From the definition of records, it is clear that last upper (lower) record is also the global maximum (minimum) of the sequence $\{X_j\}$. Hence, the statistics of last record will correspond to similar results of maximum or minimum from extreme value theory. A word of caution is warranted here, as the second last upper (lower) record will *not* be the second maximum (minimum) of the sequence. This can be easily understood by visualizing a sequence X_t whose global maximum (minimum) occurs before second maximum (minimum). As in that case, record statistics will not sense the presence of second maximum (minimum). To develop more familiarity with record sequence, let us take $\sin x, 0 \leqslant x \leq 2\pi$ as an example. In the range $[0, \pi/2]$ the upper record sequence is $\sin x$ itself and beyond that the entries will be constant 1.

Our motivations in this study are many. Firstly, while there is an elegant and simple theory of records for the case of independent random variables, we can naturally expect interesting departures for correlated processes. One such class of problems, namely random walks, have been studied in this context and remarkable departures and universalities have been uncovered. One of our motivations is to see how much of this survives for deterministic dynamical systems. For instance it has been appreciated now for more than 40 years that deterministic systems such as the kicked rotor can display normal diffusion, as well as anomalous diffusion. It is then of interest to ask if the record statistics behaves in the same manner as that of random walks in this scenario, and what happens in the different diffusion regimes. Also related question is how do the record statistics change as the system undergoes a transition to chaos. The latter question is natural here, but not so in the setting of random walks. We also wish to go on to explore quantum dynamics and review some of our recent works on record statistics. In this publication, all the material pertaining to record statistics in the classical standard map or kicked rotor is new, so also detailed derivations of the record statistics for delta correlated variables, as well the lower record statistics. Some other material, having to do with eigenfunction statistics have been partially presented in publications before (see [37]) and is included here as a brief review.

1.1. Independent and identically distributed (i.i.d.) random variables

Consider the entries of $\{X_j\}$ as being independent and identically distributed random numbers. As we have exchangeable entries, the symmetry of the sequence determines the probability of X_j being a record to be $1/j$ at the jth trial; see [32] or [1]. In other words, any of the j entries till X_j, including itself, is equally possible to be the record. Consider temperatures in a particular city, despite all the fluctuations, it is clear that it beats its own record. In formal language, the process of setting a record is persistent. Let us denote the number of records in a sequence of length n by N_n, then the above statement can be rephrased as $N_n \to \infty$ with length of sequence $n \to \infty$.

The next natural question is about the frequency with which records are broken. Define an indicator function

$$I_j = \begin{cases} 1 & \text{if there is a record at } j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Calculating the expected value of I_j is equivalent to calculating the probability of I_j taking the value 1, but this is precisely the probability of X_j being the record- which is $1/j$. Thus $\langle I_j \rangle = 1/j$. Similarly, for variance, we note that expected value of I_j^2 is the same as expected value of I_j. This immediately gives the variance as $1/j - 1/j^2$. It can be easily proven that I_j are pairwise uncorrelated (statistically independent). In other words the probability of the position of the records is a Bernoulli process, $\mathrm{Ber}(1/j)$.

As the total number of records N_n in a sequence $\{X_1, X_2, \ldots X_n\}$ is $\sum_{j=1}^n I_j$, the expectation and variance can be readily calculated:

$$\begin{aligned} \langle N_n \rangle &= \sum_{j=1}^n \langle I_j \rangle = \sum_{j=1}^n \frac{1}{j} = H_n. \\ V(N_n) &= \sum_{j=1}^n V(I_j) = \sum_{j=1}^n \frac{1}{j} - \sum_{j=1}^n \frac{1}{j^2}. \end{aligned} \tag{2}$$

Here H_n is the nth harmonic number. A remarkable, well-known fact from the theory of records is that for *i.i.d.* variables these quantities are *distribution-free*, that is independent of the particular underlying distribution $p(x)$, see [1]. For example the average number of records $\langle N_n \rangle = H_n \sim \log(n) + \gamma$, where γ is the Euler–Mascheroni constant, is indeed very small compared to the length n of the data set; typically records are rare events in independent processes. From the above it immediately follows that the variance also grows as $\log(n)$. Next section on, we will change the notation from N_n to N_R.

1.2. Random walks and records

Some of the recent works concerns the behavior of records in correlated processes, for example see [24,40,35]. One important class of such processes are random walks and the few studies on their record statistics is now very briefly reviewed. In case of random walks, x_t represents the position of a random walker at a time step, t. For discrete time steps and jump-lengths drawn from an *i.i.d.* symmetric distribution, Majumdar and Ziff showed that the probability $P(M,N)$ of M records in $t = N$ steps is given by

$$P(M,N) = \binom{2N - M + 1}{N} 2^{-2N+M-1}, \quad M \leqslant N + 1, \tag{3}$$

with mean $\langle M \rangle \sim \frac{2}{\sqrt{\pi}}\sqrt{N}$. Age statistics of record *i.e.*, how long a record survives before it is broken, is given by $\langle l \rangle \sim \frac{N}{\langle M \rangle} \sim \sqrt{\frac{\pi N}{4}}$ [24]. Thus in strong contrast to the *i.i.d.* case, the number of records is very large.