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Branching Brownian motion conditioned on particle numbers

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ABSTRACT

We study analytically the order and gap statistics of particles at time t for the one dimensional branching Brownian motion, conditioned to have a fixed number of particles at t. The dynamics of the process proceeds in continuous time where at each time step, every particle in the system either diffuses (with diffusion constant D), dies (with rate d) or splits into two independent particles (with rate b). We derive exact results for the probability distribution function of $g_k(t) = x_k(t) - x_{k+1}(t)$, the distance between successive particles, conditioned on the event that there are exactly n particles in the system at a given time t. We show that at large times these conditional distributions become stationary $P(g_k, t \to \infty|n) = p(g_k|n)$. We show that they are characterized by an exponential tail $p(g_k|n) \sim \exp\left[-\sqrt{\frac{|b-d|}{2D}}g_k\right]$ for large gaps in the subcritical ($b < d$) and supercritical ($b > d$) phases, and a power law tail $p(g_k) \sim 8\left(\frac{D}{b}\right)g_k^{-3}$ at the critical point ($b = d$), independently of n and k. Some of these results for the critical case were announced in a recent letter (Ramola et al., 2014).

1. Introduction

Branching processes are prototypical models of systems where new particles are generated at every time step – these include models of evolution, epidemic spreads and nuclear reactions amongst others [1–5]. An important model in this class is the Branching Brownian motion (BBM). We focus in this paper on the simple one-dimensional BBM, where the process starts with a single particle at the origin $x = 0$ at time $t = 0$. The dynamics proceeds in continuous time according to the following rules. In a small time interval Δt, each particle performs one of the three following microscopic moves: (i) it splits into two independent particles with probability $b\Delta t$, (ii) it dies with probability $d\Delta t$ and (iii) with the remaining probability $1 - (b + d)\Delta t$ it performs a Brownian motion moving by a stochastic distance $\Delta x(t) = \eta(t)\Delta t$. Here $\eta(t)$ is a Gaussian white noise with zero mean and delta-correlations with

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2D\delta(t_1 - t_2) \tag{1}$$

where D is the diffusion constant. The delta function in the correlator (1) can be interpreted in the following sense: when $t_1 \neq t_2$, the noise is uncorrelated. In contrast, when $t_1 = t_2 = t$, the variance $\langle \eta^2(t) \rangle = 2D/\Delta t$. A realization of the dynamics of such a process is shown in Fig. 1. Depending on the parameters b and d, the average number of particles at time t in the system exhibits different asymptotic behaviors. For $b < d$, the *subcritical* phase, the process dies and on an average there are no particles at late times. For $b > d$, the *supercritical* phase, the process is explosive and the average number of particles grows exponentially with time t. In the borderline $b = d$ case, the system is critical, where on an average there is exactly one particle in the system at all times. This critical case is relevant to several physical and biological systems with stable population distributions [4].

BBM is a paradigmatic model of branching processes with wide applications and has been studied extensively in both mathematics and physics literature [1,4,6–8]. In

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one dimension, the positions of the particles at a particular time t represent a set of random variables that are naturally ordered according to their positions on the line with $x_1(t) > x_2(t) > x_3(t) \cdots$ (see Fig. 1). It is then interesting to study their order statistics, where one is concerned with the distribution of $x_k(t)$, which denotes the position of the kth rightmost particle. An equally interesting quantity is the spacing between consecutive particles, $g_k(t) = x_k(t) - x_{k+1}(t)$ as well as the density of the particles near the tip of the branching process [9–11]. The questions related to the extremes in this one-dimensional BBM have been studied extensively over the last few decades [4,7–10,12–14]. More recently, extreme statistics in this system have found new applications in the context of estimating the perimeter and area of the convex hull of two-dimensional epidemic spreads [5].

Indeed BBM is a useful toy model to study the broader question of extreme value statistics (EVS) of correlated random variables, a field that has been growing in prominence in recent years. Several important properties sensitive to rare events can be characterized by EVS in a wide variety of disordered systems [15–17]. Although probability distributions functions (PDFs) of the extreme values of uncorrelated variables are well understood [18], the computation of extreme and near-extreme value distributions for strongly correlated variables constitute important open problems in this field [19,20]. Random walks and Brownian motion have recently proved to be useful laboratories where several exact results concerning EVS of correlated variables can be obtained [11,20,21]. In this context BBM represents a useful model where the relevant random variables (the particle positions at time t) are strongly correlated, and yet exact results concerning the extremes can be obtained. In a recent Letter [11] we briefly discussed some of these results for the critical $b = d$ case. The purpose of the present paper is twofold: (i) to provide a detailed derivation of these exact results for the critical case and (ii) to extend these results to off-critical cases $b \neq d$.

In the supercritical regime ($b > d$), the statistics of the position of the rightmost particle $x_1(t)$ has been studied

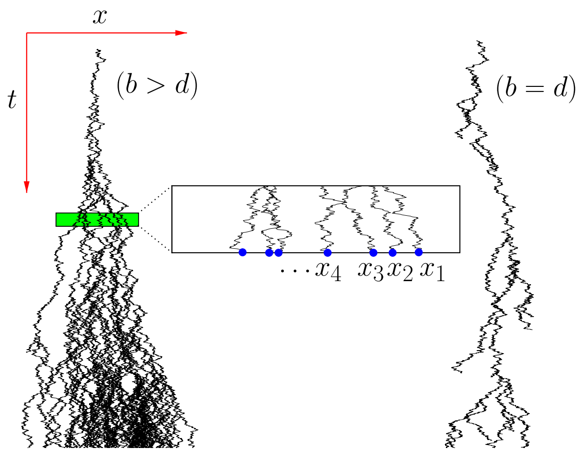

Fig. 1. A realization of the dynamics of branching Brownian motion with death (left) in the supercritical regime ($b > d$) and (right) in the critical regime ($b = d$). The particles are numbered sequentially from right to left as shown in the inset.

for a long time [7,8]. In particular, for the case $d = 0$, the cumulative distribution of $x_1(t)$ is known to be governed by the Fisher–Kolmogorov–Petrovskii–Piscounov (FKPP) equation [1,22]. This equation exhibits a traveling front solution: the average position of the rightmost particle increases linearly with time $\langle x_1(t)\rangle \sim vt$ with a constant velocity v while the width of the front remains of $\mathcal{O}(1)$ at late times. Very recently, Brunet and Derrida studied (still for $d = 0$) the order statistics, i.e., the statistics of the positions of the second, third, etc $x_2(t), x_3(t)\ldots$. They found that, while $x_k(t) \sim vt$ at late times, with the same speed v for all k, the distributions of the gaps $g_k(t)$ become independent of t for large t, while retaining a non-trivial k-dependence [9,10]. They also computed the PDF of the first gap $g_1(t)$ numerically to very high precision and also provided an argument for the observed exponentially decaying tail. Several natural questions remain outstanding. For instance, can one calculate the gap distributions for arbitrary k for $d = 0$ as well as for arbitrary b and d?

As mentioned earlier, in a recent Letter, we were able to compute the order and the gap statistics of BBM at the critical point $b = d$ at a fixed time t, by conditioning the process to have a given number of particles at time t [11]. As we will demonstrate in this paper, this method of conditioning allows us to circumvent the technical difficulties arising from the inherent non-linearities of the problem and provides exact results for arbitrary b and d. Let us briefly summarize our main results. Upon conditioning the system to have exactly n particles at time t, we derive an exact backward Fokker–Planck (BFP) equation for the joint distributions of the ordered positions of the n particles at time t. These equations can, in principle, be solved recursively for all n and the asymptotic results at late times for any fixed n can be extracted explicitly. We find that at large times, and for all b and d, the PDFs of the positions x_k's behave diffusively, $P(x_k, t \to \infty|n) \to \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x_k^2}{4Dt}\right)$, with $k = 1, 2, \ldots$. Note that for $b > d$, this diffusive behavior is in contrast with the case without conditioning on the particle number where it is ballistic. However, as in the case without conditioning, the PDFs of the gaps $g_k(t)$ become stationary in the long time limit. Moreover we show that the stationary gap PDF has an exponential tail in the super-critical ($b > d$) and sub-critical ($b < d$) regimes and an algebraic tail with exponent -3 at the critical point ($b = d$). We argue that these asymptotic tails are *universal* in the sense that they are independent of both n (the particle number) and k (the label of the gap). We also discuss the qualitative differences between the conditioned and unconditioned BBM processes.

The paper is organized as follows. In Section 2, we first compute the mean number of particles at time t after which we show in Section 3 how to compute the statistics of the rightmost particle using a BFP approach. In Section 4, we generalize the BFP approach to compute the (conditional) gap statistics between the two rightmost particles, first in the two-particle sector ($n = 2$), and then for an arbitrary number of particles $n \geqslant 2$. In Section 5, we present an asymptotic analysis of the PDF of the first gap for any n, which we then generalize to the kth gap. In Section 6, we present a comparison of our analytical results with Monte Carlo simulations, before we conclude in Section 7.