Journal of Geometry and Physics 95 (2015) 74–95

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Spinor representation of Lorentzian surfaces in $\mathbb{R}^{2,2}$

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ARTICLE INFO

Article history:
Received 27 October 2014
Received in revised form 14 April 2015
Accepted 5 May 2015
Available online 12 May 2015

MSC:
53B25
53C27
53C42
53C50

Keywords:
Lorentzian surfaces
Dirac operator
Isometric immersions
Weierstrass representation

ABSTRACT

We prove that an isometric immersion of a simply connected Lorentzian surface in $\mathbb{R}^{2,2}$ is equivalent to a normalised spinor field solution of a Dirac equation on the surface. Using the quaternions and the Lorentz numbers, we also obtain an explicit representation formula of the immersion in terms of the spinor field. We then apply the representation formula in $\mathbb{R}^{2,2}$ to give a new spinor representation formula for Lorentzian surfaces in 3-dimensional Minkowski space. Finally, we apply the representation formula to the local description of the flat Lorentzian surfaces with flat normal bundle and regular Gauss map in $\mathbb{R}^{2,2}$, and show that these surfaces locally depend on four real functions of one real variable, or on one holomorphic function together with two real functions of one real variable, depending on the sign of a natural invariant.

Introduction

Let $\mathbb{R}^{2,2}$ be the space $\mathbb{R}^4$ endowed with the metric of signature $(2, 2)$

$$g = -dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2.$$

A surface $M \subset \mathbb{R}^{2,2}$ is said to be Lorentzian if the metric g induces on M a Lorentzian metric, i.e. a metric of signature $(1, 1)$: the tangent and the normal bundles of M are then equipped with fibre Lorentzian metrics. The purpose of the paper is to study the spinor representation of the Lorentzian surfaces in $\mathbb{R}^{2,2}$; the main result is the following: if M is an abstract Lorentzian surface, E is a bundle of rank 2 on M, with a Lorentzian fibre metric and a compatible connection, and $\vec{H} \in \Gamma(E)$ is a section of E, then an isometric immersion of M into $\mathbb{R}^{2,2}$, with normal bundle E and mean curvature vector $\vec{H}$, is equivalent to a normalised section $\varphi \in \Gamma(\Sigma)$, solution of a Dirac equation $D\varphi = \vec{H} \cdot \varphi$ on the surface, where $\Sigma = \Sigma E \otimes \Sigma M$ is the spinor bundle of M twisted by the spinor bundle of E and D is a natural Dirac operator acting on Σ (we assume that spin structures are given on TM and E). We moreover define a natural closed 1-form ξ in terms of φ, with values in $\mathbb{R}^{2,2}$, such that $F := \int \xi$ is the immersion. As a first application of this representation, we derive an easy proof of the fundamental theorem of the theory of Lorentzian surfaces immersed in $\mathbb{R}^{2,2}$: a symmetric bilinear map $B : TM \times TM \to E$ is the second fundamental form of an immersion of M into $\mathbb{R}^{2,2}$ if and only if it satisfies the equations of Gauss, Codazzi and Ricci. We then deduce from the general representation in $\mathbb{R}^{2,2}$ spinor representations for Lorentzian surfaces in 3-dimensional Minkowski

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http://dx.doi.org/10.1016/j.geomphys.2015.05.002
0393-0440/

spaces $\mathbb{R}^{1,2}$ and $\mathbb{R}^{2,1}$, and also obtain new explicit representation formulas; the representations appear to be simpler than the representations obtained before by M.-A. Lawn [1,2] and by M.-A. Lawn and J. Roth [3], since only one spinor field is involved in the formulas. Our last application concerns the flat Lorentzian surfaces with flat normal bundle and regular Gauss map in $\mathbb{R}^{2,2}$: the general spinor representation formula permits to study their local structure; they locally depend on four real functions of one real variable if a natural invariant Δ is positive, and on one holomorphic function together with two real functions of one real variable if Δ is negative.

We note that a spinor representation for surfaces in 4-dimensional pseudo-Riemannian spaces already appeared in [4]; the representation formula obtained in that paper seems to be different, since the normal bundle and the Clifford action are not explicitly involved in the formula.

We quote the following related papers: the spinor representation of surfaces in $\mathbb{R}^3$ was studied by many authors, especially by Th. Friedrich in [5], who interpreted a spinor field representing a surface in $\mathbb{R}^3$ as a constant spinor field of $\mathbb{R}^3$ restricted to the surface; following this approach, the spinor representation of Lorentzian surfaces in 3-dimensional Minkowski space was studied by M.-A. Lawn [1,2] and M.-A. Lawn and J. Roth [3]. M.-A. Lawn, J. Roth and the first author then studied the spinor representation of surfaces in 4-dimensional Riemannian space forms in [6], and the first author the spinor representation of spacelike surfaces in 4-dimensional Minkowski space in [7]. Recently, P. Romon and J. Roth studied in [8] the relation between this abstract approach and more explicit representation formulas existing in the literature for surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$. Finally, the local description of the flat surfaces with flat normal bundle and regular Gauss map in 4-dimensional Euclidean and Minkowski spaces was studied in [9].

The outline of the paper is as follows: the first section is devoted to preliminaries concerning the Clifford algebra of $\mathbb{R}^{2,2}$, the spin representation, and the spin geometry of Lorentzian surfaces in $\mathbb{R}^{2,2}$. We use quaternions and Lorentz numbers to obtain concise formulas. Section 2 is devoted to the spinor representation formula of Lorentzian surfaces in $\mathbb{R}^{2,2}$. We indicate at the end of the section how to obtain the representation formulas for surfaces in $\mathbb{R}^{1,2}$ and $\mathbb{R}^{2,1}$. We then apply the representation formula to the local description of the flat Lorentzian surfaces with flat normal bundle and regular Gauss map in Section 3. An Appendix ends the paper.

1. Preliminaries

1.1. Clifford algebra of $\mathbb{R}^{2,2}$ and the spin representation

Let us denote by (e_0, e_1, e_2, e_3) the canonical basis of $\mathbb{R}^{2,2}$. The norm of a vector $x = (x_0, x_1, x_2, x_3)$ belonging to $\mathbb{R}^{2,2}$ is

$$\langle x, x\rangle := -x_0^2 + x_1^2 - x_2^2 + x_3^2.$$

To describe the Clifford algebra of $\mathbb{R}^{2,2}$, it will be convenient to consider the Lorentz numbers

$$\mathcal{A} = \{u + \sigma v : u, v \in \mathbb{R}\},$$

where σ is a formal element such that $\sigma^2 = 1$, the complexified Lorentz numbers

$$\mathcal{A}_{\mathbb{C}} := \mathcal{A} \otimes \mathbb{C} \simeq \{u + \sigma v : u, v \in \mathbb{C}\},$$

and the quaternions with coefficients in $\mathcal{A}_{\mathbb{C}}$

$$\mathbb{H}^{\mathcal{A}_{\mathbb{C}}} := \{\zeta_0 1 + \zeta_1 I + \zeta_2 J + \zeta_3 K : \zeta_0, \zeta_1, \zeta_2, \zeta_3 \in \mathcal{A}_{\mathbb{C}}\},$$

where I, J and K are such that

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -JI = K.$$

If $a = u + \sigma v$ belongs to $\mathcal{A}_{\mathbb{C}}$, we denote $\widehat{a} := u - \sigma v$, and set, for all $\zeta = \zeta_0 1 + \zeta_1 I + \zeta_2 J + \zeta_3 K$ belonging to $\mathbb{H}^{\mathcal{A}_{\mathbb{C}}}$,

$$\widehat{\zeta} := \widehat{\zeta_0} 1 + \widehat{\zeta_1} I + \widehat{\zeta_2} J + \widehat{\zeta_3} K.$$

If $\mathbb{H}^{\mathcal{A}_{\mathbb{C}}}(2)$ stands for the set of 2×2 matrices with entries belonging to $\mathbb{H}^{\mathcal{A}_{\mathbb{C}}}$, the map

$$\begin{aligned} \gamma : \quad \mathbb{R}^{2,2} &\longrightarrow \mathbb{H}^{\mathcal{A}_{\mathbb{C}}}(2) \\ (x_0, x_1, x_2, x_3) &\longmapsto \begin{pmatrix} 0 & \sigma i x_0 1 + x_1 I + i x_2 J + x_3 K \\ -\sigma i x_0 1 + x_1 I + i x_2 J + x_3 K & 0 \end{pmatrix} \end{aligned} \tag{1}$$

is a Clifford map, that is satisfies

$$\gamma(x)^2 = -\langle x, x\rangle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all $x \in \mathbb{R}^{2,2}$, and thus identifies

$$Cl(2,2) \simeq \left\{ \begin{pmatrix} p & q \\ \widehat{q} & \widehat{p} \end{pmatrix} : p \in \mathbb{H}_0, \ q \in \mathbb{H}_1 \right\}, \tag{2}$$