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Banach–Lie algebroids associated to the groupoid of partially invertible elements of a W^*-algebra

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ABSTRACT

In the paper we study the algebroid $\mathcal{A}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements over the lattice $\mathcal{L}(\mathfrak{M})$ of orthogonal projections of a W^*-algebra $\mathfrak{M}$. In particular the complex Banach manifold structure of these objects is investigated. The expressions on the algebroid Lie brackets for $\mathcal{A}(\mathfrak{M})$ and related algebroids are given in noncommutative operator coordinates in the explicit way. We also describe structure of the groupoid of partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and the frame groupoid $\mathcal{G}^{lin}\mathcal{A}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ as well as their algebroids.

1. Introduction

Nowadays the theory of Lie groupoids and Lie algebroids is an important and invaluable part of contemporary differential geometry. One can consider Lie groupoids and Lie algebroids as a framework for the investigation of symmetry of objects having fibre bundle structure. They also are a natural generalizations of Lie groups and Lie algebras, respectively. In the monograph [1] of Mackenzie, whose personal contribution to the subject is notable, one finds compact presentation of the subject as well as historical references at the end of each chapter.

While the concept of groupoid in topology [2] and Lie groupoids and Lie algebroids in differential geometry [3–5] appeared in 1960s and 1970s of the last century the growth of interest to these notions was truly inspired by problems of mathematical physics. It was firstly Poisson geometry, where after the paper of Karasev [6] and seminar note of Coste, Dazord and Weinstein [7] the symplectic realization of Poisson manifold by symplectic groupoid was defined and investigated. Then it was observed that the existence of a Poisson structure on a manifold M induces a Lie algebroid structure on its cotangent bundle T^*M, and many Poisson constructions are more easily carried out in terms of this Lie algebroid. Subsequently many other researchers, first of all Weinstein with his collaborators used groupoid and algebroid methods in mechanics and in problems of quantization, see Cannas da Silva, Weinstein [8], Connes [9] and references of therein.

Admitting some modifications of the basic definitions one can investigate the above mentioned structures in the infinite dimensional case, i.e. in the framework of the category of smooth Banach manifolds. An example of the investigations of Banach Lie algebroids one finds in [10]. However, a crucial difficulty appears when the modelling Banach space does not

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have Schauder basis, e.g. it happens for Banach algebra of bounded operators $L^\infty(\mathcal{H})$ of Hilbert space $\mathcal{H}$. The reason is that in this case does not exist coordinate description of the investigated structures, in particular case the Poisson structure.

Nevertheless the category of W^*-algebras (von Neumann algebras) is the one which generates a rich class of Banach–Lie groupoids and Banach–Lie algebroids as well as the related class of Banach Poisson manifolds which have nice properties and could be handled in the operator coordinate manner which is strictly associated to W^*-algebra structure. We were motivated in our investigations by importance of von Neumann algebras in quantum physics and the fact that the predual $\mathfrak{M}_*$ of the W^*-algebra $\mathfrak{M}$ has canonically defined structure of Banach Lie Poisson space (see [11]) and moreover the precotangent bundle $T_*G(\mathfrak{M})$ of the Banach Lie group $G(\mathfrak{M})$ of invertible elements of $\mathfrak{M}$ is a weak symplectic realization of $\mathfrak{M}_*$.

The main place in our considerations is occupied by the Banach Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of $\mathfrak{M}$ with the lattice of orthogonal projections $\mathcal{L}(\mathfrak{M})$ as the base manifold which was defined and described in [12].

In Section 2 we investigate the family of locally trivial transitive subgroupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$, parametrized by $p_0 \in \mathcal{L}(\mathfrak{M})$. Namely, we define a complex analytic atlas on $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ consistent with its groupoid structure, i.e. we show that $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is a complex analytic groupoid.

In Section 3 we describe the Banach–Lie algebroid $\mathcal{A}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$. Our description is based on the groupoid isomorphism given in Proposition 3.2, which shows that one can consider the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ as the gauge groupoid of the G_0-principal bundle with $P_0 := \mathcal{G}(\mathfrak{M})_{p_0}$ as the total space and the group $G(p_0\mathfrak{M}p_0)$ of invertible elements of $p_0\mathfrak{M}p_0$ denoted by G_0. Among others we present coordinate expressions (61) and (69), see also (82) and (84), for Lie algebroid bracket of sections of $\mathcal{A}_{p_0}(\mathfrak{M})$. In this section we present detailed description of Banach–Lie groupoid $\mathcal{U}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ of partial isometries as well as the corresponding algebroid $\mathcal{A}^u_{p_0}(\mathfrak{M})$, see Proposition 3.3 and Proposition 3.4.

As an example we consider in Section 4 the algebroid of the frame groupoid of tautological vector bundle $\mathbb{E} \to G(N, \mathcal{H})$ over Grassmannian of N-dimensional subspace of the Hilbert space $\mathcal{H}$.

In Section 5 we discuss various groupoids associated in a canonical way to $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ including the frame groupoid $\mathcal{G}^{lin}\mathcal{A}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ of the algebroid $\mathcal{A}_{p_0}(\mathfrak{M})$.

Finally Section 6 contains the description of the algebroid $\mathcal{A}^{lin}_{p_0}(\mathfrak{M})$ which is the algebroid of the groupoid $\mathcal{G}^{lin}\mathcal{A}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ proving that $\mathcal{A}^{lin}_{p_0}(\mathfrak{M})$ is a subalgebroid of the algebroid $\mathfrak{D}(TP_0)$ of derivations of $\Gamma^\infty(TP_0)$.

At the end the following two facts are worth to be noted. In the case $\mathfrak{M} = L^\infty(\mathcal{H})$ one can consider $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $\mathcal{U}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $\mathcal{A}_{p_0}(\mathfrak{M})$, $\mathcal{A}^u_{p_0}(\mathfrak{M})$, P_0 and P_0^u as universal objects in the corresponding categories. However, we will not discuss this question, leaving it for subsequent paper. The groupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ as well as the corresponding algebroids $\mathcal{A}_{p_0}(\mathfrak{M})$ provide interesting examples of the complex analytic Banach–Lie groupoids and algebroids, respectively.

2. Groupoid of partially invertible elements of W^*-algebra

Such a class of Banach–Lie groupoids was introduced and investigated in [12]. Here we recall some necessary notions and statements concerning the subject. By definition the groupoid $\mathcal{G}(\mathfrak{M})$ of partially invertible elements of W^*-algebra $\mathfrak{M}$ will consist of such elements $x \in \mathfrak{M}$ for which $|x| = (x^*x)^{\frac{1}{2}}$ is an invertible element of the W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$, where p is the support of $|x|$. We have natural maps $l : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ and $r : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M})$ on the complete lattice $\mathcal{L}(\mathfrak{M})$ of orthogonal projections of the W^*-algebra $\mathfrak{M}$, defined as the left and right supports of $x \in \mathcal{G}(\mathfrak{M})$, respectively. Taking $\mathcal{L}(\mathfrak{M})$ as the base set of $\mathcal{G}(\mathfrak{M})$ one can identify l with the target map, r with the source map and inclusion $\varepsilon : \mathcal{L}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ with the object inclusion map. The partial multiplication of $x, y \in \mathcal{G}(\mathfrak{M})$ is the algebraic product in $\mathfrak{M}$. Note that $\mathcal{G}(\mathfrak{M}) \subset \mathfrak{M}$ and $xy \in \mathcal{G}(\mathfrak{M})$ if $r(x) = l(y)$. The two sided inversion x^{-1} of $x \in \mathcal{G}(\mathfrak{M})$ is defined by its polar decomposition (see [13])

$$x = u|x| \tag{1}$$

in the following way

$$x^{-1} = |x|^{-1}u^*. \tag{2}$$

One easily verifies that the above maps and operations define groupoid structure $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on the set $\mathcal{G}(\mathfrak{M})$ over the lattice $\mathcal{L}(\mathfrak{M})$.

Proposition 2.1. *An element $x \in \mathfrak{M}$ of W^*-algebra $\mathfrak{M}$ belongs to $\mathcal{G}(\mathfrak{M})$ if and only if there exists $y \in \mathfrak{M}$ such that*

$$yx = r(x) \quad \text{and} \quad xy = l(x) \tag{3}$$

and one has $l(y) = r(x)$ and $r(y) = l(x)$. The element y belongs to $\mathcal{G}(\mathfrak{M})$ and is defined uniquely by $x \in \mathcal{G}(\mathfrak{M})$.

Proof. If $x \in \mathcal{G}(\mathfrak{M})$ then $y = |x|^{-1}u^*$ satisfies relations (3). Now, let $y \in \mathfrak{M}$ satisfy conditions (3). Then

$$u^*u = r(x) = yx = yu|x| \tag{4}$$