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Higher order generalized Euler characteristics and generating series

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ABSTRACT

For a complex quasi-projective manifold with a finite group action, we define higher order generalized Euler characteristics with values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. We compute the generating series of generalized Euler characteristics of a fixed order of the Cartesian products of the manifold with the wreath product actions on them.

Let X be a topological space (good enough, say, a quasi-projective variety) with an action of a finite group G. For a subgroup H of G, let $X^H = \{x \in X : Hx = x\}$ be the fixed point set of H. The orbifold Euler characteristic $\chi^{orb}(X, G)$ of the G-space X is defined, e.g., in [1,2]:

$$\chi^{orb}(X, G) = \frac{1}{|G|} \sum_{\substack{(g_0, g_1) \in G \times G: \\ g_0 g_1 = g_1 g_0}} \chi(X^{\langle g_0, g_1 \rangle}) = \sum_{[g] \in G_*} \chi(X^{\langle g \rangle}/C_G(g)), \quad (1)$$

where G_* is the set of conjugacy classes of elements of G, $C_G(g) = \{h \in G : h^{-1}gh = g\}$ is the centralizer of g, and $\langle g \rangle$ and $\langle g_0, g_1 \rangle$ are the subgroups generated by the corresponding elements.

The higher order Euler characteristics of (X, G) (alongside with some other generalizations) were defined in [3,4].

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Definition. The *Euler characteristic* $\chi^{(k)}(X,G)$ *of order* k of the G-space X is

$$\chi^{(k)}(X,G)=\frac{1}{|G|}\sum_{\substack{\mathbf{g}\in G^{k+1}:\\ g_ig_j=g_jg_i}}\chi(X^{\langle\mathbf{g}\rangle})=\sum_{[g]\in G_*}\chi^{(k-1)}(X^{\langle g\rangle},C_G(g)),\tag{2}$$

where $\mathbf{g}=(g_0,g_1,\ldots,g_k)$, $\langle\mathbf{g}\rangle$ is the subgroup generated by $g_0,g_1,\ldots,g_k$, and $\chi^{(0)}(X,G)$ is defined as $\chi(X/G)$.

The usual orbifold Euler characteristic $\chi^{orb}(X,G)$ is the Euler characteristic of order 1, $\chi^{(1)}(X,G)$.

The higher order generalized Euler characteristics take values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck ring of complex quasi-projective varieties. This is the abelian group generated by the isomorphism classes $[X]$ of quasi-projective varieties modulo the relation:

– if Y is a Zariski closed subvariety of X, then $[X]=[Y]+[X\setminus Y]$.

The multiplication in $K_0(\mathrm{Var}_{\mathbb{C}})$ is defined by the Cartesian product. The class $[X]$ of a variety X is the universal additive invariant of quasi-projective varieties and can be regarded as a generalized Euler characteristic of X. Let $\mathbb{L}$ be the class $[\mathbb{A}^1_{\mathbb{C}}]$ of the affine line and let $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ be the extension of the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ by all the rational powers of $\mathbb{L}$.

The formula for the generating series of the generalized orbifold Euler characteristics of the pairs (X^n,G_n) in [5] uses the (natural) power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ (and over $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$) defined in [6]. (See also [7] and [5] for some generalizations of this concept.) This means that for a power series $A(T)\in 1+t\cdot R[[t]]$ ($R=K_0(\mathrm{Var}_{\mathbb{C}})$ or $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$) and for an element $m\in R$ there is defined a series $(A(T))^m\in 1+t\cdot R[[t]]$ so that all the properties of the exponential function hold. For a quasi-projective variety M, the series $(1-t)^{-[M]}$ is the Kapranov zeta-function of M: $\zeta_{[M]}(t):=(1-t)^{-[M]}=1+[M]\cdot t+[\mathrm{Sym}^2 M]\cdot t^2+[\mathrm{Sym}^3 M]\cdot t^3+\cdots$, where $\mathrm{Sym}^k M=M^k/S_k$ is the kth symmetric power of the variety M. A geometric description of the power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is given in [6] or [5]. The (natural) power structures over $K_0(\mathrm{Var}_{\mathbb{C}})$ and over $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ possess the following properties:

(1) $(A(t^s))^m=(A(t))^m|_{t\mapsto t^s}$;
(2) $(A(\mathbb{L}^s t))^m=(A(t))^{\mathbb{L}^s m}$.

One can define a power structure over the ring $\mathbb{Z}[u_1,\ldots,u_r]$ of polynomials in r variables with integer coefficients in the following way. Let $P(u_1,\ldots,u_r)=\sum_{\underline{k}\in\mathbb{Z}^r_{\geq 0}}p_{\underline{k}}\underline{u}^{\underline{k}}\in\mathbb{Z}[u_1,\ldots,u_r]$, where $\underline{k}=(k_1,\ldots,k_r)$, $\underline{u}=(u_1,\ldots,u_r)$, $\underline{u}^{\underline{k}}=u_1^{k_1}\cdot\ldots\cdot u_r^{k_r}$, $p_{\underline{k}}\in\mathbb{Z}$. Define

$$(1-t)^{-P(u_1,\ldots,u_r)}:=\prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}}(1-\underline{u}^{\underline{k}}t)^{-p_{\underline{k}}},$$

where the power (with an integer exponent $-p_{\underline{k}}$) means the usual one. This gives a λ-structure on the ring $\mathbb{Z}[u_1,\ldots,u_r]$ and therefore a power structure over it (see, e.g., [5, Proposition 1])

i.e., for polynomials $A_i(\underline{u})$, $i\geq 1$, and $M(\underline{u})$, there is defined a series $\left(1+A_1(\underline{u})t+A_2(\underline{u})t^2+\cdots\right)^{M(\underline{u})}$ with the coefficients from $\mathbb{Z}[u_1,\ldots,u_r]$.

Let $r=2$, $u_1=u$, $u_2=v$. Let $e:K_0(\mathrm{Var}_{\mathbb{C}})\to\mathbb{Z}[u,v]$ be the ring homomorphism which sends the class $[X]$ of a quasi-projective variety X to its Hodge–Deligne polynomial $e(X;u,v)=\sum h_X^{ij}(-u)^i(-v)^j$.

Remark. Let R_1 and R_2 be rings with power structures over them. A ring homomorphism $\varphi:R_1\to R_2$ induces the natural homomorphism $R_1[[t]]\to R_2[[t]]$ (also denoted φ) by $\varphi\left(\sum a_i t^i\right)=\sum\varphi(a_i)\,t^i$. In [5, Proposition 2], it was shown that if a ring homomorphism $\varphi:R_1\to R_2$ is such that $(1-t)^{-\varphi(m)}=\varphi\left((1-t)^{-m}\right)$ for any $m\in R$, then $\varphi\left((A(t))^m\right)=(\varphi(A(t)))^{\varphi(m)}$ for $A(\underline{t})\in 1+tR[[t]]$, $m\in R$.

There are two natural homomorphisms from the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u,v]$ of polynomials in two variables: the Euler characteristic (with compact support) $\chi:K_0(\mathrm{Var}_{\mathbb{C}})\to\mathbb{Z}$ and the Hodge–Deligne polynomial. Both possess the following well known identities:

(1) the formula of I.G. Macdonald [8]:

$$\chi(1+[X]t+[\mathrm{Sym}^2 X]t^2+[\mathrm{Sym}^3 X]t^3+\cdots)=(1-t)^{-\chi(X)},$$

(2) and the corresponding formula for the Hodge–Deligne polynomial (see [9, Proposition 1.2]):

$$e(1+[X]t+[\mathrm{Sym}^2 X]t^2+\cdots)=(1-T)^{-e(X;u,v)}=\prod_{p,q}\left(\frac{1}{1-u^pv^qt}\right)^{e^{p,q}(X)}.$$

These properties and the previous remark imply that the corresponding homomorphisms respect the power structures over the corresponding rings: $K_0(\mathrm{Var}_{\mathbb{C}})$ and $\mathbb{Z}[u,v]$ respectively, see [7].