Chaos, Solitons & Fractals 73 (2015) 1–9

Contents lists available at ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Alternate superior Julia sets

Anju Yadav *, Mamta Rani

Department of Computer Science, Central University of Rajasthan, Kishangarh, India

A R T I C L E I N F O

Article history:
Received 24 July 2014
Accepted 12 December 2014
Available online 15 January 2015

A B S T R A C T

Alternate Julia sets have been studied in Picard iterative procedures. The purpose of this paper is to study the quadratic and cubic maps using superior iterates to obtain Julia sets with different alternate structures. Analytically, graphically and computationally it has been shown that alternate superior Julia sets can be connected, disconnected and totally disconnected, and also fattier than the corresponding alternate Julia sets. A few examples have been studied by applying different type of alternate structures.

1. Introduction

The natural systems do not evolve according to a unique dynamics as there are many different interactions. To understand the evolution of a natural process, the alternated iterations of different dynamics may be used. Danca et al. [7] were first to study the numerical and analytical connectedness properties of the alternated Julia sets. They applied alternations to the pairs of maps of complex quadratic family and showed that the Julia sets of alternated iterations can be connected, totally disconnected as well as disconnected also. Danca et al. [6] showed graphical exploration of connectivity of alternated Julia sets and also explained the set of disconnected (not totally disconnected) alternate Julia sets.

Parrondo gave a paradox, a concept of alternate iterations, which has been applied on two real Mandelbrot maps [1]. Later on, it was applied on different problems in various ways [3–5,9,25]. The Parrondo's paradox has also been applied to study the different forms of logistic map [10,13,14,21,23,24].

Fatou [8] and Julia [11] in 1918–1919 proved a result named Fatou–Julia theorem. In 1992, Branner and Hubbard [2] stated a conjecture in Fatou–Julia theorem, which was later on proved by Qiu and Yin [15]. Hence it completes Fatou-Julia theorem as follows [*cf.* 7]:

Theorem 1

(i) *The Julia set is connected if and only if all the critical orbits are bounded.*
(ii) *The Julia set is totally disconnected, a cantor set, if (but not only if) all the critical orbits are unbounded.*
(iii) *For a polynomial with at least one critical orbit unbounded, the Julia set is totally disconnected if and only if all the bounded critical orbits are aperiodic* [*cf.* 7].

According to Part (i), (ii) and (iii) a Julia set can be connected, totally disconnected and also disconnected.

Definition 1.1 (*Disconnected Julia set*). A Julia set is disconnected if it can be separated into open and disjoint sets such that neither set is empty, and both sets are combined to give the original set [*cf.* 6].

Definition 1.2 (*Totally disconnected Julia set*). The Julia set X is totally disconnected, if the connected components in X are one point set [*cf.* 6].

Rani and Kumar [20], see also [16], used superior iterates to generate superior Julia sets. To study the details of superior fractal models, one may refer to Singh, Mishra and Sinkala [22]. To compute superior Julia sets for complex

* Corresponding author. Tel.: +91 9460887735.
E-mail address: anju.anju.yadav@gmail.com (A. Yadav).

http://dx.doi.org/10.1016/j.chaos.2014.12.008
0960-0779/

polynomials, Rani and Kumar [16,20] gave new and considerably improved escape criterions and called the same as superior escape criterions.

When a function is iterated in superior orbit, more points form bounded sequences than that of Picard orbit. Therefore, superior Julia sets are fattier than their corresponding Julia sets [12]. In 2010, Rani [17,18] computed tricorns, multicorns, and antiJulia sets in superior orbit for the antipolynomial $z \to \bar{z}^d + c$, where $z \in$ Complex plane, $(d \in N) \geqslant 2$ and $\bar{z}$ is conjugate of z.

In this paper, different types of alternations have been applied on quadratic and cubic superior Julia sets, and showed that the filled superior Julia sets are connected, disconnected and totally disconnected. In Section 2, we have discussed the basic parameters and equations that we have taken into our account for the study. In Section 3, we have given the description of experimental approach and theorems to solve the proposed problem. In Section 4, we showed the results obtained by our approach followed by concluding remarks in Section 5.

2. Preliminaries

In this section, we give preliminary definitions that form the basis of our work.

Definition 2.1 (*Alternated Julia set*). Let us consider the family of quadratic maps $P_c : z_{n+1} = z_n^2 + c$. The alternated filled Julia set $K_{c_1c_2}$ is the set of points of complex plane whose orbits are bounded when the system is iterated alternatively as follows:

$$P_{c_1c_2} : z_{n+1} = \begin{cases} z_n^2 + c_1, & \text{if } n \text{ is even,} \\ z_n^2 + c_2, & \text{if } n \text{ is odd,} \end{cases}$$

where z, c, c_1, $c_2 \in C$. C is a complex plane [7].

Definition 2.2 (*Peano-Picard orbit*). Let X be a non-empty metric space and $f : X \to X$. For an initial point x_0 in X, the Picard orbit (generally called orbit of f or trajectory of f) is the set of all iterates of a point x_0, that is:

$$O(f, x_0) := \{x_n : x_n = f(x_{n-1}), \quad n = 1, 2, \ldots\}$$

Notice that the orbit $O(f, x_0)$ of f at the initial point x_0 is $\{f_n(x_0)\}$. In all that follows, by orbit we mean Picard orbit unless otherwise stated [20].

The above mentioned complex polynomials have been iterated in O alternatively by Danca et al. [6,7]. They showed that the alternate Julia sets may not only be connected or totally disconnected but disconnected as well.

Definition 2.3 (*Superior iterates*). Let A be a subset of real numbers and $f : A \to A$ For $x_0 \in A$, construct a sequence $\{x_n\}$ in the following manner:

$$x_1 = \beta_1 f(x_0) + (1 - \beta_1)x_0$$
$$x_2 = \beta_2 f(x_1) + (1 - \beta_2)x_1;$$
$$\ldots$$
$$x_n = \beta_n f(x_{n-1}) + (1 - \beta_n)x_{n-1};$$

where $0 < \beta_n \leqslant 1$ and $\{\beta_n\}$ is convergent when its value is away from 0 [16,19,20].

Definition 2.4 (*Superior orbit*). The sequence $\{x_n\}$ constructed above is called Mann sequence of iterates or superior sequence of iterates. We denote it by $SO(f, x_0, \beta_n)$. Notice that $SO(f, x_0, \beta_n)$ with $\beta_n = 1$ is $O(f, x_0)$ [16,19,20].

Definition 2.5 (*Superior Julia set*). The set of points whose orbits are bounded under SO of a function $Q(z)$ will be called the filled superior Julia set, denoted by SJ. Superior Julia set of Q is the boundary of filled superior Julia set [20].

Theorem 2 (*General escape criterion for SJ sets*). *Suppose we have* $|z_k| > \max\left\{|c|, \left(\frac{2}{\beta}\right)^{\frac{1}{k-1}}\right\}$ *for some* $k \geqslant 0$, $\lambda > 0$. *Then* $|z_{k+1}| > (1 + \lambda)|z_k|$, *so* $|z_n| \to \infty$ *as* $n \to \infty$ [20].

3. Proposed algorithm

In this section, quadratic and cubic complex polynomials have been considered to generate alternate *SJ* sets. Therefore, we divide our study into two sections; quadratic and cubic.

3.1. Quadratic polynomial

Let us consider $P_{c_1} : z_{n+1} = z_n^2 + c_1$ and $P_{c_2} : z_{n+1} = z_n^2 + c_2$ be the two families of quadratic maps. $P_{c_1c_2}$ in superior orbit is defined as:

$$P_{c_1c_2} : z_{n+1} = \begin{cases} \beta_n(z_n^2 + c_1) + (1 - \beta_n)z_n & \text{if } n \text{ is even} \\ \beta_n(z_n^2 + c_2) + (1 - \beta_n)z_n & \text{if } n \text{ is odd} \end{cases} \tag{3.1}$$

where $0 < \beta_n \leqslant 1$ and $\{\beta_n\}$ is convergent when its value is away from 0.

For the sake of simplicity, we have considered $\beta_1, \beta_2, \ldots, \beta_n = \beta$ for n iterations in whole paper. The orbit generated by odd and even iterates of $P_{c_1c_2}$ in SO with initial point z_0 is as follows:

$$z_0,$$
$$z_1 = \beta(z_0^2 + c_1) + (1 - \beta)z_0,$$
$$z_2 = [\beta((\beta(z_0^2 + c_1) + (1 - \beta)z_0)^2 + c_2)] + [(1 - \beta)(\beta(z_0^2 + c_1) + (1 - \beta)z_0)],$$
$$\begin{aligned} z_3 = {} & [\beta([\beta((\beta(z_0^2 + c_1) + (1 - \beta)z_0)^2 + c_2)] \\ & + [(1 - \beta)(\beta(z_0^2 + c_1) + (1 - \beta)z_0)])^2 + c_1] \\ & + [(1 - \beta)([\beta((\beta(z_0^2 + c_1) + (1 - \beta)z_0)^2 + c_2)] \\ & + [(1 - \beta)(\beta(z_0^2 + c_1) + (1 - \beta)z_0)])], \end{aligned}$$
$$\ldots$$

It responds to general expressions for $i = 1, 2, 3\ldots$ as follows:

$$z_{2i-1} = \beta\left(z_{2i-2}^2 + c_1\right) + (1 - \beta)z_{2i-2}, \tag{3.2}$$

$$z_{2i} = \beta\left(z_{2i-1}^2 + c_2\right) + (1 - \beta)z_{2i-1}, \tag{3.3}$$

$$z_{2i+1} = \beta\left(z_{2i}^2 + c_1\right) + (1 - \beta)z_{2i}. \tag{3.4}$$

We take the auxiliary complex quadratic polynomial $Q_{c_1c_2}$ in SO as it is difficult to study the connectivity of alternated SJ sets of $P_{c_1c_2}$ in SO. $Q_{c_1c_2}$ is defined as

$$Q_{c_1c_2} : z_{n+1}^* = \beta\left(\left(z_n^{*2} + c_1\right)^2 + c_2\right) + (1 - \beta)\left(z_n^{*2} + c_1\right).$$