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Formula for Fibonacci sequence with arbitrary initial numbers

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ABSTRACT

In this paper the formula for Fibonacci sequences with arbitrary initial numbers has been established by using damped oscillation equation. The formula has an exponential and an oscillatory part, it does not separate the indexes of odd and even members of the series and it is applicable on the continual domain. With elementary conditions the formula is reduced to Lucas series, and the square of Lucas series has a catalytic role in the relation of hyperbolic and trigonometric cosine. A complex function is given and the length of Fibonacci spiral is calculated. Natural phenomena support the validity of the proposed concept.

1. Introduction

The discovery and use of the constant φ = 1,6180339 is attributed to Phiadis (500–432 BC), the greatest sculptor of classical Greece and architect of Pharthenos on Acropolis around 447 BC. Euclid (325–265 BC), gave the first recorded definition of "extreme and mean ratio" for which it was later established to have ratio of the constant φ. Leonardo Fibonacci (1170–1240) promoted his series but never found out its most important characteristic, convergence of successive members of constant φ. Fra Luca Pacioli (1445–1517) in the period from 1496 to 1498 wrote a book "*De divina proportione*". In 1497 in Milan he teaches math to Leonardo da Vinci (1452–1519) who drew five illustrations for his book. The influence of Divine proportions on Leonardo's further work is well known. Michael Maestlin (1550–1631), professor at the University of Tübingen, in 1597 published the first known approximation of the φ constant. Johannes Kepler (1571–1630) while still a student of Michael Maestlin, is the first who proved that the constant φ is the limiting value of quotient of successive numbers in Fibonacci sequence. The following two centuries the constant φ carries the name "Kepler's limit". Charles Bonnet (1720–1793) Swiss philosopher and botanist, was the first to point out that spiral plants (*phyllotaxis*) in their structure contain the proportion of the constant φ. This determined the importance of the φ constant within natural phenomena.

So far it hasn't been established who determined identicalness between geometrical proportion of Euclidean definition of "extreme and mean ratio"and "Kepler's limit" successive Fibonacci numbers [1].·It is presumed that it was Martin Ohm (1792–1872) German mathematician and younger brother of famous physicist Georg Ohm. In the book "*Pure Elementary Mathematics*" from 1835 Martin Ohm introduced the term "*golden section*" (*goldener Schnitt*) for the first time. French mathematician and physicist Jacques Philippe Marie Binet (1786–1856) determined reoccurring patterns in the Fibonacci sequence. However, over this result there is still doubt in Binet's originality. Some historical sources claim that Abraham de Moavre (1667–1754) was the first one to determine these patterns, almost half a century before Binet. Édouard Lucas (1842–1891) French mathematician gave the Fibonacci sequence its name by which it goes today. Édouard Lucas also determined special series with initial numbers 2 and 1. Lucas expanded Fibonacci sequence to negative whole numbers. Binet's formulas are by simple transformations applied to Lucas series as well.

Significant improvements to Binet's formulas was done by Stakhov and Rozin [2]. Apart from previous presumptions about continuality of the domain, relation has

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between Fibonacci and Lucas sequence with hyperbolic functions been determined. However, the determined mathematical concept still created odd and even indexes, by the use of hyperbolic sine and hyperbolic cosine. The problem of alternative mark sign which separated members of Fibonacci sequence with odd and even index was eliminated recently, with a simple and effective idea, with the use of trigonometric cosine [3]:

$$F(n) = \frac{\varphi^n - \varphi^{-n}\cos(n\pi)}{\sqrt{5}} \tag{1}$$

The imperative question is if a general formula can be determined which for arbitrary initial numbers which determines members of the sequence on the principles determined by Fibonacci, on a continual domain.

2. Fibonacci sequence and damped oscillations

First we recall the equation for damped oscillations:

$$x(n) = \rho e^{\beta n} cos(\omega n + \omega_0) \tag{2}$$

Coefficient β in usual conditions is equal to quotient $b/2m$, where b is coefficient of proportionality of speed of movement of a body in a resistant environment. Based on the coefficient of proportionality, the force of resistance is equal to the product of coefficient b multiplied by the speed of movement. The value of m comes from the (real) oscillator equation, which moves along the abscissa, thus it follows:

$$ma_x = -kx - bv \tag{3}$$

The well known Binet's formula for calculation of Fibonacci numbers:

$$F(n) = \frac{1}{\sqrt{5}} \cdot \frac{\varphi^{2n} - (-1)^n}{\varphi^n} = \frac{1}{\sqrt{5}} \cdot \left(\varphi^n - \frac{1}{\varphi^n}(-1)^n\right) \tag{4}$$

expanded with the use of cosine based on expression (1) divided into the exponential part and damped oscillations part:

$$F(n) = \frac{\varphi^n}{\sqrt{5}} - \frac{1}{\varphi^n\sqrt{5}}\cos(n\pi) \tag{5}$$

The damped oscillations part is actually expression (2) with $\beta = \ln\varphi$ and $\rho = \frac{1}{\sqrt{5}}$.

Thus, we rewrite the equation for calculation of the Fibonacci sequence on a continuous domain:

$$\begin{aligned} F(n) &= \frac{1}{\sqrt{5}} \cdot \left(\varphi^n - e^{-n\ln\varphi}\cos(n\pi)\right) \\ &= \frac{\varphi^n}{\sqrt{5}} - \frac{e^{-n\ln\varphi}\cos(n\pi)}{\sqrt{5}} \end{aligned} \tag{6}$$

By analyzing function (6), it can be noticed that the basic axis of the series is defined as exponential function of the golden section constant, and the value of the Fibonacci sequence oscillate in damped manner around the exponential axis (Fig. 1). Function (6) is also defined for negative values of the independent variable.

The proposed concept is not something new. However, here it is necessary to explicitly state the analogy withdamped oscillations, due to the coefficient of damping

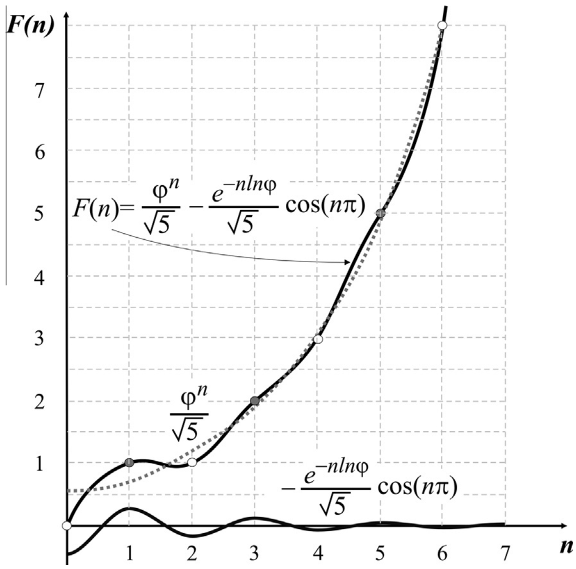

Fig. 1. Function of Fibonacci sequence in continuous domain with decomposition on exponential and oscillating parts.

$\beta = ln\varphi$, which will apart from the constant π play a crucial role in the calculation of the length of the Fibonacci spiral. The value of natural logarithm of the golden section constant has been established multiple times in the existing literature [2,4,5].

3. Formula for Fibonacci sequence with arbitrary initial numbers

If we mark with $F_{(a,b)}(n)$ Fibonacci sequence with initial numbers (a,b):

$$\begin{aligned} &F_{(a,b)}(0) = a, \quad F_{(a,b)}(1) = b, \\ &F_{(a,b)}(n) + F_{(a,b)}(n+1) = F_{(a,b)}(n+2) \end{aligned} \tag{7}$$

Members of this sequence are: a, b, $a + b$, $a + 2b$, $2a + 3b$, $3a + 5b$, $5a + 8b$,.... Sequence $F_{(a,b)}(n)$ can be decomposed in two sequences (Fig. 2):

which form Fibonacci sequence with initial numbers (a, b):

$$F_{(a,b)}(n) = F_{(a,0)}(n) + F_{(0,b)}(n) \tag{8}$$

Where the following identities are valid:

$$F_{(a,0)}(n) = a \cdot F_{(1,0)}(n); \quad F_{(0,b)}(n) = b \cdot F_{(0,1)}(n) \tag{9}$$

The series $F_{(1,0)}(n)$ is actually a translated series of $F_{(0,1)}(n)$ i.e.:

$$F_{(1,0)}(n) = F_{(0,1)}(n-1) \tag{10}$$

$$F_{(1,0)}(n) = F_{(0,1)}(n-1) = \frac{\varphi^{n-1}}{\sqrt{5}} - \frac{e^{-(n-1)\ln\varphi}\cos((n-1)\pi)}{\sqrt{5}} \tag{11}$$

$$F_{(1,0)}(n) = \frac{\varphi^{n-1}}{\sqrt{5}} - \frac{\varphi e^{-n\ln\varphi}(\sin(n\pi)\sin\pi + \cos(n\pi)\cos\pi)}{\sqrt{5}} \tag{12}$$