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Modified homogeneous balance method: Applications and new solutions

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ABSTRACT

This work is devoted to application of the modified homogeneous balance method to obtain generalized bilinear forms of some well-known soliton equations: the Korteweg de Vries equation, the scalar Boussinesq equation and the Kaup–Boussinesq equations. These bilinear forms are solved for new solutions using the perturbation method and the principle of superposition.

1. Introduction

The soliton equations play an important role particularly in fluid mechanics and generally in mathematical physics. The most well-known prototype of this class of equations is the Korteweg–de Vries (KdV) equation [1]. This equation was abandoned for a long time since it was not clear at the time if a solitary wave could exist and how it should appear. Only until the seventies of the last century, the KdV equation attracted attention of physicists and mathematicians due to the exact integration for soliton solution by the Hirota's bilinear method [2] and almost the same time by the inverse scattering transform [3]. The name soliton is coined to reflect the particle-like behavior of such solutions, and henceforth the suffix -on follows. Both methods were subsequently and successfully applied to other soliton equations such as the modified Korteweg–de Vries equation [4] and the sine–Gordon equation [5,6]. The latter finds its application in different branches of physics [7,8]. For importance of these equations, the respective modulation theories, which provide an asymptotic mathematical tool, were developed in [9–11] using the variational asymptotic procedure whose theory can be found in [12,13]. Since the discovery of exact solutions of KdV equation, a quest for exact solutions of other equations of the same class has arisen. Several approaches to nonlinear wave equations have been recently proposed, for example homogeneous balance method [14–17], *F*-expansion method [18], tanh method [19], variational iteration method [20–22], exp-function method [23,24], sin-cosine function method [25,26], and so forth. All fore-mentioned methods possess simultaneously advantages and disadvantages towards various particular equations. They help to find the solutions in a direct way, but at the same time, do not provide a systematic procedure to obtain solutions of higher order like Hirota's bilinear method. Recently, Liu found that there is a link between the bilinear form and the homogeneous balance method. This remarkable finding led him to the modified homogeneous balance method whose steps are adequately described in [17]. Nevertheless, the bilinear forms found in there can be still generalized and so can their corresponding solutions. In this paper, we present an application of modified homogeneous balance method to seek for the generalized bilinear forms of two soliton equations and one system of equations followed in the list.

- Korteweg de Vries equation,
- Scalar Boussinesq equation,
- Kaup–Boussinesq equations.

A crucial test of soundness of these forms is to find their exact solutions. Beyond these results, we present a few new interesting solutions of the obtained equations.

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2. Korteweg–de Vries equation

2.1. Derivation of bilinear form

Due to the significantly historical importance, we start first with a well-known prototype of soliton equation, KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

Following [2], we propose the solution Ansatz

$$u(x,t) = \varphi(\theta(x,t))_{xx} + a, \quad \varphi(\theta) = 2\log\theta. \tag{2}$$

Substituting this Ansatz into Eq. (1), we obtain

$$A\theta_x\varphi_{\theta\theta\theta} + C\varphi_{\theta\theta} + B_x\varphi_\theta = 0, \tag{3}$$

where the coefficients depending on function θ are given by

$$\begin{aligned}
A &= 6a\theta_x^2 + \theta_t\theta_x + 4\theta_x\theta_{xxx} - 3\theta_{xx}^2,\\
B &= 6a\theta_{xx} + \theta_{xt} + \theta_{xxxx},\\
C &= 18a\theta_x\theta_{xx} - 2\theta_{xx}\theta_{xxx} + 5\theta_x\theta_{xxxx} + 2\theta_x\theta_{xt} + \theta_t\theta_{xx}.
\end{aligned}$$

It can be proved that $C = A_x + B\theta_x$ so that (3) is rewritten in the form

$$A\theta_x\varphi_{\theta\theta\theta} + A_x\varphi_{\theta\theta} + B\theta_x\varphi_{\theta\theta} + B_x\varphi_\theta = 0 \quad \text{or} \quad (A\varphi_{\theta\theta} + B\varphi_\theta)_x = 0.$$

An integration of this equation yields

$$A\varphi_{\theta\theta} + B\varphi_\theta = q(t),$$

where $q(t)$ is a function depending only on time variable. Using $(2)_2$, this equation can be given in a more explicit form

$$B\theta - A = 6a(\theta\theta_{xx} - \theta_x^2) + \theta_{xt}\theta - \theta_x\theta_t + \theta\theta_{xxxx} - 4\theta_x\theta_{xxx} + 3\theta_{xx}^2 = \theta^2\frac{q(t)}{2},$$

which is engendered in a more elegant form by using the Hirota's bilinear differential operator[1]

$$D_x(D_t + 6aD_x + D_x^3)\theta\cdot\theta = \theta^2 q(t). \tag{4}$$

2.2. Solution of the bilinear form equation

In special case $q = 0$, the bilinear form (4) can be solved for n-soliton by following the perturbation method [27]. Since the procedure of finding n-soliton solution based on bilinear form is standardized, we suppress here the rigorous verification, which might be nothing else but repetition of works presented in [2,4,5], and provide only the final result

$$\begin{aligned}
u(x,t) &= 2\frac{\theta\theta_{xx} - \theta_x^2}{\theta^2} + a,\\
\theta &= 1 + \sum_{n=1}^{N}\sum_{C_N^n}\left[\prod_{k<l}^{(n)}\gamma(i_k,i_l)\right]\exp(\eta_{i_1} + \cdots + \eta_{i_n}),\\
\eta_i &= k_ix + \omega_it + \delta_i, \quad \omega_i = -6ak_i - k_i^3,\\
\gamma(i,j) &= -\frac{(k_i-k_j)\left[\omega_i - \omega_j + 6a(k_i-k_j) + (k_i-k_j)^3\right]}{(k_i+k_j)\left[\omega_i + \omega_j + 6a(k_i+k_j) + (k_i+k_j)^3\right]} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}.
\end{aligned}$$

[1] In this work, we adopt the denotation of the bilinear differential operator defined by Hirota in [27].

In this solution formula, C_N^n indicates all possible combination of n elements from the set of N elements

$$\Omega_N = \{j \in \mathbb{N}, 1 \leqslant j \leqslant N\},$$

and $\prod_{k<l}^{(n)}$ is the product of all possible combinations of these taken n elements. Note that the solution provided here is different from what we have known due to the appearance of an arbitrary constant a.

Discussion. It is interesting that the n-soliton solution provided above gives more flexibility in determining the velocity of each soliton which is characterized by

$$c_i = -\omega_i/k_i = 6a + k_i^2.$$

Thus, one soliton propagates to the left if $6a + k_i^2 < 0$, to the right if $6a + k_i^2 > 0$, and stands still otherwise. To illustrate this argument, we pick up a 3-soliton solution generated with the following parameters: $k_1 = 1$, $k_2 = 2, k_3 = 3, a = -2/3$. Accordingly, the velocities of three respective solitons are $c_1 = -3, c_2 = 0, c_3 = 5$, so two of them propagate in opposite directions whereas the other does not propagate at all. In Fig. 1, such exact solution is plotted at three different time instants to make the explanation realizable.

3. Scalar Boussinesq equation

3.1. Derivation of bilinear form

We consider now the scalar Boussinesq equation

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0. \tag{5}$$

Following [28], we derive the bilinear form of this equation using the Ansatz

$$u(x,t) = \varphi(\theta(x,t))_{xx} + a, \quad \varphi(\theta) = 6\log\theta. \tag{6}$$

Substituting this Ansatz into Eq. (5), it is expanded to

$$A\theta_x^2\varphi_\theta^{(4)} + C\varphi_{\theta\theta\theta} + D\varphi_{\theta\theta} + B_{xx}\varphi_\theta = 0, \tag{7}$$

where the coefficients of derivatives of φ are given by

$$\begin{aligned}
A &= \theta_t^2 + 2a\theta_x^2 - 3\theta_{xx}^2 + 4\theta_x\theta_{xxx},\\
C &= \theta_x^2\theta_{tt} + 4\theta_t\theta_x\theta_{xt} + \theta_t^2\theta_{xx} + 12a\theta_x^2\theta_{xx} - 3\theta_{xx}^3 + 9\theta_x^2\theta_{xxxx},\\
D &= 2\theta_{xt}^2 + \theta_{tt}\theta_{xx} + 6a\theta_{xx}^2 + 2\theta_t\theta_{xxt} + 2\theta_x\theta_{xtt} + 3\theta_{xx}\theta_{xxxx}\\
&\quad + 8a\theta_x\theta_{xxx} + 6\theta_x\theta_{xxxxx} - 2\theta_{xxx}^2,\\
B &= \theta_{tt} + 2a\theta_{xx} + \theta_{xxxx}.
\end{aligned}$$

Direct calculation shows that

$$C = A\theta_{xx} + 2A_x\theta_x + B\theta_x^2, \quad D = A_{xx} + 2B_x\theta_x + B\theta_{xx}.$$

Introducing a new dependent variable $\psi = \varphi_\theta$, substituting the above relations into (7) and applying the chain rule of differentiation, we find that (7) can be transformed to

$$(A\psi_\theta + B\psi)_{xx} = 0.$$

Integrating this equation with respect to x twice, and using the definition $\psi = 6/\theta$, it is explicitly written in the form

$$\begin{aligned}
&\theta\theta_{tt} - \theta_t^2 + 2a(\theta\theta_{xx} - \theta_x^2) + \theta\theta_{xxxx} - 4\theta_x\theta_{xxx} + 3\theta_{xx}^2\\
&\quad = \frac{1}{2}\theta^2(p(t)x + q(t)).
\end{aligned}$$