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Subharmonic and homoclinic solutions for second order Hamiltonian systems with new superquadratic conditions ☆

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ABSTRACT

The existence of infinitely many subharmonic solutions is obtained for a class of nonautonomous second order Hamiltonian systems with a new superquadratic condition. Furthermore, we can get the existence of homoclinic solutions as the limit of subharmonics under a stronger superquadratic condition which is still weaker than the growth conditions in the references.

1. Introduction and main results

In this paper, we consider the existence of subharmonic and homoclinic solutions for the following second order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in R, \tag{1}$$

where $W \in C^1(R \times R^N, R)$ and L is a continuous T-periodic matrix valued function for all $t \in [0, T]$. A kT-periodic solution of problem (1) for some positive integer k is called to be subharmonic. Furthermore, a solution $u(t)$ of problem (1) is homoclinic (to 0) if $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. Moreover, if $u(t) \neq 0, u(t)$ is called a nontrivial homoclinic solution. Here and subsequently, $\nabla W(t, x)$ denotes the gradient with respect to the x variable, and $(\cdot, \cdot) : R^N \times R^N \to R$ denotes the standard inner product in R^N, moreover, $|\cdot|$ is the induced norm.

The homoclinic orbits are important in study of the behavior of dynamical systems which have been researched since Poincaré. In last decades, the existence and multiplicity of homoclinic orbits have been intensively studied by many mathematicians with variational methods [1–24,26–31,35–37] and the reference therein.

In 1990, Rabinowitz in [17] obtained the following theorem:

Theorem A (*See [17]*). *Suppose that W is T-periodic in t satisfying*

(A_1) *there exists a constant $\theta > 2$ such that*

$$0 < \theta W(t, x) \leqslant (\nabla W(t, x), x)$$

for every $t \in R$ and $x \in R^N \setminus \{0\}$;

(A_2) $\nabla W(t, x) = o(|x|)$ as $|x| \to 0$ uniformly for $t \in R$.

(A_3) *L is a positive definite symmetric matrix valued function.*

Then problem (1) possesses a nontrivial homoclinic solution.

In [17], the author proved the existence of at least one nontrivial homoclinic solutions for problem (1) as the limit of the subharmonic solutions which are obtained by the Mountain Pass Theorem.

With the similar method, Izydorek and Janczewska in [10], Tang and Xiao in [23] generalized Theorem A by replacing $L(t)u(t)$ with a general form. But in both of these two papers, the authors needed the growth condition (A_1) which plays an important role in the proof. This condition is well known as the global Ambrosetti–Rabinowitz

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condition which can help prove the compact condition. In recent years, there are many papers [7,8,28,30,31,35] obtained the existence and multiplicity of homoclinic solutions of problem (1) with some other superquadratic conditions on W instead of (A_1). Subsequently, we set

$$\widetilde{W}(t,x) = (\nabla W(t,x), x) - 2W(t,x).$$

In 2008, Ding and Lee [7] considered system (1) with W satisfying the following conditions

(W_0) $W \in C^2(R \times R^N, R)$ and $\nabla W(t,0) = 0$ and $\nabla^2 W(t,0) = 0$ for all $t \in R$.
(W_1) $W(t,0) = 0$ for all $t \in R$.
(W_2) $\widetilde{W}(t,x) > 0$ for all $t \in R, x \in R^N \setminus \{0\}$.
(W_3) there exist $\tau \in (0,1)$ and $R_1, a_0 > 0$ such that

$$\widetilde{W}(t,x) \geqslant a_0 \frac{(\nabla W(t,x), x)}{|x|^{2-\tau}} \quad \text{for all} \quad t \in R \quad \text{and} \quad |x| \geqslant R_1.$$

(W_4) $W(t,x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in $t \in R$.
We can deduce from (W_2), (W_3) and (W_4) that.
(W_5) $\frac{\widetilde{W}(t,x)}{W(t,x)} |x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in $t \in R$.
Moreover, we can see that condition (W_5) implies that.
(W_6) *there are constants* $a_1, r_\infty > 0$ *such that*

$$\widetilde{W}(t,x) \geqslant a_1 \frac{W(t,x)}{|x|^2} \quad \text{forall} \quad t \in R \quad \text{and} \quad |x| \geqslant r_\infty.$$

Condition (W_6) is introduced by Tang and Wu [25] to obtain the existence of periodic solutions for problem (1) with some other conditions.

When L and W are neither autonomous nor periodic in t, the situation is more complicated since the lack of compactness of the Sobolev embedding. In order to get the compactness back, we usually need some different conditions on L and there are many papers concerning on topic, which will not be discussed more since in this paper, we only consider the case when L is periodic in t.

In this paper, we get the existence of subharmonic solution for problem (1) by using the Mountain Pass Theorem under condition (W_6). With the stronger condition (W_5), we prove that the subharmonic solutions converge to a nontrivial homoclinic solution by some uniform estimates. First of all, we state our existence result of subharmonic solutions for problem (1).

Theorem 1.1. *Suppose that* W *and* L *are* T*-periodic with respect to* $t, T > 0$ *satisfying* $(A_2), (A_3), (W_1), (W_4), (W_6)$ *and the following condition*

(W_2') $\widetilde{W}(t,x) \geqslant 0$ for all $t \in R$ and $x \in R^N$.
Then there exists a sequence $\{k_i\} \subset N$, $k_i \to \infty$, *and corresponding distinct* $2k_iT$ *periodic solutions of problem* (1).

Remark 1. The conditions of Theorem 1.1 are different from the results in the references. For example, let

$$W(t,x) = \begin{cases} q_1|x|^6 + q_2|x|^4 & \text{for } |x| \leqslant \sqrt{3\pi} \\ |x|^2\ln(1+|x|^2) + \sin|x|^2 - \ln^2(1+|x|^2) & \text{for } |x| \geqslant \sqrt{3\pi}, \end{cases} \tag{2}$$

where

$$q_1 = (3\pi)^{-3}\left(2\ln^2(1+3\pi) - 3\pi\ln(1+3\pi) - \frac{3\pi + 6\pi\ln(1+3\pi)}{1+3\pi}\right),$$

$$q_2 = (3\pi)^{-2}\left(6\pi\ln(1+3\pi) - 3\ln^2(1+3\pi) + \frac{3\pi + 6\pi\ln(1+3\pi)}{1+3\pi}\right).$$

Then we have

$$\liminf_{|x|\to\infty} \frac{(\nabla W(t,x), x) - 2W(t,x)}{|x|^p} = 0 \tag{3}$$

for any $p > 0$. We can see that (2) satisfies the conditions of Theorem 1.1 but not the results in [26,32–34].

If we replace the condition (W_6) by the stronger condition (W_5), we can get the existence of at least one homoclinic orbit for problem (1) which is the following theorem.

Theorem 1.2. *Suppose that* W *and* L *are* T*-periodic with respect to* $t, T > 0$ *satisfying* $(A_2), (A_3), (W_1), (W_2), (W_4), (W_5)$. *Then problem* (1) *possesses at least one nontrivial homoclinic solution* $u \in W^{1,2}(R, R^N)$.

Remark 2. It is easy to see that (W_2), (W_3) and (W_4) can imply (W_5) which can also be obtained by (A_1). Furthermore, condition (A_2) can be deduced by (W_0) and (W_1). Then Theorem 1.2 generalizes Theorem A and Theorem 1.2 in [7]. And we can set the following example

$$W(t,x) = \begin{cases} b_1|x|^6 + b_2|x|^8 & \text{for } |x| \leq 8\sqrt{\pi} \\ |x|^2\ln^{\frac{3}{2}}(1+|x|^{\frac{4}{3}}) + \ln^{\frac{1}{2}}(1+|x|^{\frac{4}{3}})\cos|x|^2 - \ln^2(1+|x|^{\frac{4}{3}}) & \text{for } |x| \geqslant 8\sqrt{\pi}, \end{cases} \tag{4}$$

where $b_1 = 4(8\sqrt{\pi})^{-6}(\vartheta_1 - \sqrt{\pi}\vartheta_2)$, $b_2 = 2^{-1}(8\sqrt{\pi})^{-8}(8\sqrt{\pi}\vartheta_2 - 6\vartheta_1)$ with $\vartheta_1 = 64\pi A_1^{\frac{3}{2}} + A_1^{\frac{1}{2}} - A_1^2$, $\vartheta_2 = 16\sqrt{\pi}A_1^{\frac{3}{2}} + 96\pi A_1^{\frac{1}{2}}A_2 + \frac{1}{2}A_1^{-\frac{1}{2}}A_2 - 2A_1A_2$ and $A_1 = \ln(1 + 16\pi^{\frac{2}{3}})$, $A_2 = \frac{\frac{8}{3}\pi^{\frac{1}{6}}}{1+16\pi^{\frac{2}{3}}}$. It is easy to see that (4) satisfies the conditions of Theorem 1.2, but not the results in [8,13,21,27–31] since (4) satisfies (3).

2. Proof of Theorem 1.1

For each $k \in N$, let $E_k = W^{1,2}_{2kT}(R, R^N)$ be the Hilbert space of $2kT$ periodic functions under the following norm

$$\|u\|_{E_k} := \left(\int_{-kT}^{kT} |\dot{u}(t)|^2 dt + \int_{-kT}^{kT} (L(t)u(t), u(t)) dt\right)^{1/2}. \tag{5}$$

Set

$$L^p_{2kT}(R, R^N) = \left\{u : [-kT, kT] \to R^N \,|\, \|u\|_{L^p_{2kT}(R,R^N)} < \infty\right\},$$

where