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Review

Theta vocabulary I

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ABSTRACT

This paper is an annotated list of transformation properties and identities satisfied by the four theta functions θ_1, θ_2, θ_3, θ_4 of one complex variable, presented in a ready-to-use form. An attempt is made to reveal a pattern behind various identities for the theta-functions. It is shown that all possible 3, 4 and 5-term identities of degree four emerge as algebraic consequences of the six fundamental bilinear 3-term identities connecting the theta-functions with modular parameters τ and 2τ.

Contents

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1. Foreword

The theta functions introduced by Jacobi [1] (see also [2–5]) are doubly (quasi)periodic analogues of the basic trigonometric functions $\sin(\pi u)$ and $\cos(\pi u)$. Let the two (quasi)periods be 1 and $\tau \in \mathbb{C}$ with the condition $\Im\tau > 0$. The basic theta functions are $\theta_1(u|\tau)$, $\theta_2(u|\tau)$, $\theta_3(u|\tau)$, $\theta_4(u|\tau)$. The theory of theta functions is a sort of "elliptically deformed" trigonometry. In essence the functions sin and cos are the same because $\cos x = \sin(x + \frac{\pi}{2})$, but everybody knows that in practice it is more convenient to work with the two functions rather than one. Likewise, the four theta functions can be obtained from any one of them by simple transformations like shifts of the argument and multiplying by a common factor, but it is more convenient to deal with the set of four instead of one.

The "elliptic deformation" of the trigonometric functions may go in two ways depending on which property of the former one wants to preserve or generalize. One is a deformation in the class of *entire functions* (the north-east arrow in the diagram below). It leads to the quasi-periodic theta functions, which are regular functions in the whole complex plane. The other one is in the class of *doubly periodic functions*. The (infinite) second period of the trigonometric functions becomes finite (equal to τ) at the price of breaking the global analyticity, so the elliptic functions sn, cn and dn, which are doubly periodic analogues of trigonometric sin and cos are *meromorphic functions* in the complex plane.

$$
\begin{array}{ccc}
 & & \{\theta_1, \theta_2, \theta_3, \theta_4\} \\
 & \nearrow & \\
\{\sin,\ \cos\} & & \\
 & \searrow & \\
 & & \{\mathrm{sn},\ \mathrm{cn},\ \mathrm{dn}\}
\end{array} .
$$

In fact the basic elliptic functions are constructed as ratios of the theta functions and in this sense the latter seem to be more fundamental.

In practical calculations with trigonometric functions (and their hyperbolic cousins), one needs just a few identities for the basic functions sin and cos like the addition formula $\sin(x + y) = \sin x \cos y + \sin y \cos x$. It is not difficult to remember them all or derive any forgotten one from scratch using the definitions $\sin x = -i(e^{ix} - e^{-ix})/2$, $\cos x = (e^{ix} + e^{-ix})/2$. For the theta functions, the situation is much more involved. They are connected by a plethora of identities most of which are not obvious, not suitable for memorizing and cannot be derived from scratch in any easy way. Here is what Mumford wrote in Chapter 1 of his book "Tata lectures on Theta I" [5] after presenting a list of ponderous identities for theta functions:

> *"We have listed these at such length to illustrate a key point in the theory of theta functions: the symmetry of the situation generates rapidly an overwhelming number of formulae, which do not however make a completely elementary pattern. To obtain a clear picture of the algebraic implications of these formulae altogether is then not usually easy".*

All this is aggravated by the fact that there are several different systems of notation for theta functions in use.

In the present paper we make an attempt to bring some order into this conglomeration of formulae. We show that the 3, 4 and 5-term identities of degree four (i.e. with products of four theta functions in each term), referred to as Weierstrass addition formulae, Jacobi relations, and Riemann identities, respectively, can be obtained by purely algebraic manipulations from *six basic* 3*-term theta relations of degree two* connecting theta functions with modular parameters τ and 2τ. Starting with the six "elementary bricks", it is possible to derive 52 fundamental relations of degree four containing four independent variables. Besides, we give the complete list of all important particular identities which are appropriate specifications of the basic bilinear and degree four ones. Recently, Koornwinder has proved [6] that the Weierstrass addition formulae and the Riemann identities are equivalent. We reproduce this result in a very simple way.

In the future we plan to address more specific questions related to the role of theta functions in the theory of integrable systems and lattice models of statistical mechanics.

2. Theta functions

2.1. Theta functions with characteristics

Fix the modular parameter $\tau \in \mathbb{C}$ such that $\Im\tau > 0$ and consider the infinite series [5]:

$$\theta_{a,b}(u|\tau) = \sum_{k\in\mathbb{Z}} \exp\Big\{\pi i\tau(k+a)^2 + 2\pi i(k+a)(u+b)\Big\}, \tag{2.1}$$

where $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$. The series is absolutely convergent for any $u \in \mathbb{C}$ and defines the entire function $\theta_{a,b}(u|\tau)$. It is called the theta function with characteristics a, b. These functions are connected by the relations