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Classification of left-invariant metrics on the Heisenberg group

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ABSTRACT

In this study, we investigate the Riemannian and Lorentzian geometry of left-invariant metrics on the Heisenberg group H_{2n+1}, of dimension $2n + 1$. We describe the space of all the left-invariant metrics of Riemannian and Lorentzian signatures up to automorphisms of the Heisenberg group. Thus, we classify quadratic forms of the corresponding signatures with respect to the action of the symplectic group. We also investigate the curvature properties and holonomy of these metrics. The most interesting is the Lorentzian metric with a parallel, null, central, left-invariant vector field. Rahmani proved that this metric is flat in the case of Heisenberg group H_3. We show that this metric is not flat in higher dimensions.

0. Introduction

Research into left-invariant Riemannian metrics on Lie groups began with the classical work of Milnor (see [1]), who mainly investigated the curvature properties of such metrics on various types of Lie groups, e.g., three-dimensional, unimodular, and compact.

The approach used in the present study is to classify all of the left-invariant metrics on a particular Lie group G, or on some class of Lie groups. In [2], Lauret showed that if a non-abelian Lie group has a unique left-invariant Riemannian metric up to homothety, then it is isomorphic to either the hyperbolic space H^n, or the direct product $\mathbb{R}^{n-3} \times H_3$ of the abelian group and the Heisenberg group H_3. Therefore, most Lie groups are expected to have more than one left-invariant metric.

Left-invariant metrics on four-dimensional nilpotent Lie groups were classified in [3] for the Riemannian case and in [4] for Lorentzian metrics. In [5], the authors classified all of the four-dimensional Lie groups with the Einstein and Ricci flat left-invariant Lorentzian metric. In general, there are striking differences between the Riemannian and pseudo-Riemannian cases. For example, up to homothety, there is a unique left-invariant Riemannian metric on the Heisenberg group H_3, whereas there are three metrics in the Lorentzian case, as shown by Rahmani [6]. These three metrics are distinguished by the characteristics of the center of the corresponding Lie algebra $\mathfrak{h}_3$, with respect to the induced inner product. The metric with a null center is flat.

The Heisenberg group H_{2n+1} has many application in physics and quantum mechanics, and it is well understood from the viewpoint of sub-Riemannian geometry (see [7]). The Heisenberg group H_{2n+1} with the most simple left-invariant metric is an important example of the Sasakian manifold. Although this metric was known to Sasaki [8], its relationship with the Heisenberg group was not known at that time and it was discovered only recently [9,10]. Left-invariant Lorentzian metrics on

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the three-dimensional Heisenberg group H_3 were classified in [6] and investigated further in [11]. A particular left-invariant Lorentzian metric on the Heisenberg group was studied in [12].

These findings motivated us to study left-invariant metrics on the Heisenberg group H^{2n+1} and their geometry. In the present study, we generalize the result of Rahmani [6] concerning Lorentzian metrics on the Heisenberg group H_3 to Riemannian and Lorentzian metrics on the Heisenberg group H_{2n+1}. The natural action of group $Sp(2n, \mathbb{R})$ requires a more subtle and difficult technique for classifying left-invariant metrics.

The remainder of this paper is organized as follows. In Section 1, we provide the necessary notation by defining the Heisenberg group H_{2n+1}, and describing its Lie algebra $\mathfrak{h}_{2n+1}$ and group of automorphisms $Aut(\mathfrak{h}_{2n+1})$. A left-invariant metric g on H_{2n+1} is determined by the inner product q on the Lie algebra $\mathfrak{h}_{2n+1}$. If S is the matrix of the inner product q in one basis, then $F^TSF, F \in Aut(\mathfrak{h}_{2n+1})$ is the matrix of q in another basis with the same Lie algebra commutators. Therefore, describing the inner products on $\mathfrak{h}_{2n+1}$ is equivalent to finding the classes F^TSF, $F \in Aut(\mathfrak{h}_{2n+1})$, where the symmetric matrix S is of prescribed signature. In Lemma 3.1, we prove that automorphisms of $\mathfrak{h}_{2n+1}$ are closely related to the symplectic group $Sp(2n, \mathbb{R})$. Therefore, in Section 2, we aim to find a canonical form of the symmetric matrix S of Riemannian or Lorentzian signatures under the action F^TSF, where $F \in Sp(2n, \mathbb{R})$. In some general sense, the answer to this question is given by the famous Williamson theorem (see [13]). For a positive definite matrix S, an explicit proof using a variational argument can be found in [14] and a constructive proof can be found in [15]. In Theorem 2.1, we present a proof that works for both Riemannian and Lorentzian matrices S. The eigenvalues $\pm i\sigma_j, j = 1, \dots, n$ in the Riemannian case (see Theorem 2.1) are called *symplectic eigenvalues* of matrix S. They are important symplectic invariants, which are related to rigidity theorems in symplectic geometry, such as Gromov's non-squeezing theorem (see [14] for a good overview of the problem).

In Section 3, we provide a classification of the inner products on the Heisenberg Lie algebra: Riemannian in Theorem 3.1 and Lorentzian in Theorem 3.2. The Riemannian inner products correspond to the metrics represented by the identity matrix on the so-called "non-isotropic" Heisenberg Lie groups (see [7], Chapter 4). The Lorentzian inner products give rise to left-invariant metrics on H_{2n+1}, which generalize the metrics of Rahmani [6].

Finally, in Section 4, we calculate the Ricci and scalar curvature (Theorem 4.1) and holonomy (Theorem 4.2) of the left-invariant Riemannian and Lorentzian metrics.

1. Preliminaries

Let us denote the matrix of the standard complex structure on $\mathbb{R}^{2n}$ by

$$J = J_{2n} = \mathrm{diag}(J_1, \dots, J_1),$$

where $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The standard symplectic form in vector space $\mathbb{R}^{2n}$ can be written in the form

$$\omega(x, y) = x^T J y, \quad x, y \in \mathbb{R}^{2n}.$$

Recall that the symplectic group is the group of all linear transformations of $\mathbb{R}^{2n}$ that preserve ω, i.e.,

$$Sp(2n, \mathbb{R}) := \{F \in Gl_{2n}(R) \mid \omega(Fx, Fy) = \omega(x, y)\} = \{F \mid F^T J F = J\}.$$

The Heisenberg group H_{2n+1} is defined on the base manifold $\mathbb{R}^{2n} \times \mathbb{R}$ with the multiplication

$$(x, \lambda) \cdot (y, \mu) := (x + y, \lambda + \mu + \omega(x, y)).$$

Its Lie algebra is

$$\mathfrak{h}_{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R} = \mathbb{R}^{2n} \oplus \mathcal{Z} = \{(x, \lambda) \mid x \in R^{2n}, \lambda \in \mathbb{R}\}$$

with the commutator

$$[(x, \lambda), (y, \mu)] = (0, \omega(x, y)). \tag{1}$$

Note that $\mathcal{Z} = \mathrm{span}(z)$ is the one-dimensional center of $\mathfrak{h}_{2n+1}$.

The Heisenberg group is 2-step nilpotent. Moreover, any 2-step nilpotent Lie group of odd dimension with a one-dimensional center is locally isomorphic to the Heisenberg group H_{2n+1}.

Theorem 1.1 ([16]). *The exponential map is a diffeomorphism on the nilpotent Lie group. Thus, the Heisenberg group H_{2n+1} is diffeomorphic to $\mathbb{R}^{2n+1}$.*

Denote $Aut(G)$ and $Aut(\mathfrak{g})$ as the groups of automorphisms of the Lie group and its Lie algebra, respectively, and $Aut_0(G), Aut_0(\mathfrak{g})$ as their identity components.

Theorem 1.2 ([16]). *If the Lie group is simply connected, then $Aut_0(G) \cong Aut_0(\mathfrak{g})$.*

2. Symplectic classification of Riemannian and Lorentzian inner products

In the next section, we show that the group of automorphisms of Lie algebra $\mathfrak{h}_{2n+1}$ contains the symplectic group $Sp(2n, \mathbb{R})$. Therefore, we are interested in the orbits of the action of group $Sp(2n, \mathbb{R})$ in the space of inner products.