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On the geometry of some unitary Riemann surface braid group representations and Laughlin-type wave functions

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ABSTRACT

In this note we construct the simplest unitary Riemann surface braid group representations geometrically by means of stable holomorphic vector bundles over complex tori and the prime form on Riemann surfaces. Generalised Laughlin wave functions are then introduced. The genus one case is discussed in some detail also with the help of noncommutative geometric tools, and an application of Fourier–Mukai–Nahm techniques is also given, explaining the emergence of an intriguing Riemann surface braid group duality.

1. Introduction

In this paper we study the simplest unitary representations of the braid group associated to a general Riemann surface from a geometrical standpoint. Besides being interesting in itself, such an investigation could prove useful in topological quantum computing [1,2], where unitary braid group representations are employed for constructing quantum gates (topology would then automatically enforce robustness and fault tolerance), with the Fractional Quantum Hall Effect (FQHE) possibly yielding the physical clue to its practical implementation [1].

Recall that the FQHE arises for a (Coulomb) interacting spin-polarised 2d-electron gas, at low temperature and in the presence of a strong magnetic field. It is usually observed in semiconductor structures, such as electrons trapped in a thin layer of GaAs surrounded by AlGaAs, Si-MOSFETs (see e.g. [3]) and it has been recently detected in graphene [4] as well. The ground state of such a system can be approximately (but most effectively) described by a *Laughlin wave function* of the form (in a plane geometry, [5,3]):

$$\prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^{N}|z_i|^2}. \qquad (L)$$

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Here N is the number of electrons in the sample, m is an odd integer (this ensuring Fermi statistics). One notes the appearance of the ground state of a quantum harmonic oscillator. The quantity $\nu = \frac{1}{m}$ is the *filling factor* intervening in the fractional quantisation of the *Hall conductance*:

$$\sigma_H = \nu \frac{e^2}{h}$$

and, in the limit $N \to \infty$, equals the electron density per state: $\nu = \frac{N}{N_S}$ with N_S the number of magnetic flux quanta: $N_S = B \cdot \mathcal{A}/\Phi_0$ (B is the modulus of the constant magnetic field (acting perpendicularly to the layer), $\mathcal{A}$ is the area of the given sample, whereas $\Phi_0 := hc/e$ is the flux quantum). The number N_S also gives the degeneracy of the lowest Landau level (for the free system), which appears as a degenerate ground state of a quantum harmonic oscillator.

On the mathematical side, Landau levels admit elegant algebro-geometric descriptions along the lines of geometric quantisation (see e.g. [6–8]): for instance, if the layer is a (closed) Riemann surface of genus g, the lowest Landau level is the space of holomorphic sections of a suitable holomorphic line bundle [9,10]; on a torus ($g = 1$) it can be realised as a space of theta functions, see e.g. [11,12], and also [13] and below.

Now, *on the one hand*, it turns out that the elementary excitations around the Laughlin ground state are *quasiparticles/holes* having *fractional charge* $\pm\nu e$ [5,3] and *anyon statistics* $(-1)^\nu$ [14,3], and this leads to considering the *braid group* associated to the N-point configuration space of the given layer (N now being the number of quasiparticles/holes). Wave functions for quasiparticles/holes can be cast in the form (L), with ν replacing m (see [14,3]).

On the other hand, the filling factor $\nu = 1/m$ (together with others) for a *torus* sample has been interpreted as the *slope* (that is, degree over rank) of a *stable* holomorphic vector bundle over the corresponding "spectral", or "Brillouin manifold" (which is again a torus, parametrising all admissible boundary conditions, see [15,16,11] and below); therefore, the filling factor has a *topological* meaning. (For ν integral one recovers the interpretation of the integral Quantum Hall Effect via the first Chern class of a line bundle over the Brillouin manifold, see e.g. [17,18,3].)

One of the aims of the present note – which is a greatly expanded and substantially improved version of [19] – is to show that the above coincidence has an abstract braid group theoretical origin: we consider a general closed Riemann surface – so that the role of the Brillouin manifold is played by the *Jacobian* of the surface (cf. [10]) – and its associated braid group, with the Bellingeri presentation [20]; then the equalities, in the genus one case,

$$\nu := \text{filling factor} = \text{statistical parameter} = \text{slope of a stable vector bundle}$$

can be derived from a group theoretical perspective, and can be suitably generalised. (See e.g. [21] for a recent comprehensive coverage of braid groups.)

Our first observation is that, in our context, braiding can approached via representations of the Weyl–Heisenberg group corresponding to the (rational) statistical parameter ν, both infinite dimensional and finite dimensional. Then, generalising [13], we observe that the infinite dimensional representations can be constructed geometrically on L^2-sections of holomorphic Hermitian stable bundles over the Jacobian of the Riemann surface under consideration. Stable bundles are irreducible holomorphic vector bundles over Kähler manifolds admitting a Hermitian–Einstein structure (HE) – namely a (unique) Hermitian connection with central constant curvature – in view of the Donaldson–Uhlenbeck–Yau theorem [22–25]. Specifically, the representation of the Weyl–Heisenberg group we look for stems from suitable parallel transport operators associated to the HE-connection (which will have constant curvature, essentially given by the statistical parameter ν). The solution is actually reduced to finding suitable *projectively flat HE-bundles over Jacobians*, which can be obtained via the classical Matsushima construction [26,27,25]. In particular, we get a "slope-statistics" formula $\mu = \nu g!$ (with μ denoting the slope of a holomorphic vector bundle). We also show that, at least for genera $g = 2,3$, which allow for totally split Jacobians (cf. [28]), one can take box products of bundles on elliptic curves.

The other important geometrical ingredient needed to describe the statistical behaviour governing "particle" exchange is the Klein prime form on a Riemann surface, manufactured via theta function theoretic tools. The problem of extracting general roots of a line bundle then arises and it is circumvented by exploiting a universal property of the prime form. Then we define, following Halperin [5,14,3], (Laughlin type) vector valued wave functions obeying, in general, *fractional* statistics and having their "centre of mass" part represented by holomorphic sections of the above bundles (see also [15,9,16,29]). Theorem 3.2 summarises the whole construction of the RS-braid group representations we aimed at.

The successive developments portray an interesting "braid duality", and run as follows. Focusing in particular on the $g = 1$ case, we show that everything can be made even more explicit by resorting to A. Connes' noncommutative geometric setting [30,31] for noncommutative tori and to the notion of noncommutative theta vector introduced by A. Schwarz [32], encompassing the classical notions. The upshot is that the "centre of mass" parts of Laughlin wave functions are precisely the Schwarz theta vectors. A notable feature is now the following: the space of theta vectors naturally determines a finite dimensional braid group representation corresponding to the reciprocal parameter $\nu' = 1/\nu$, which, via Matsushima, gives rise to a projectively flat HE-bundle with the corresponding slope. Therefore a (Matsushima–Connes (MC)) "duality" emerges. We shall then prove that this duality is essentially the one given by the Fourier–Mukai–Nahm (FMN) transform. In particular, the noncommutative theta vector approach will be used to calculate the Nahm-transformed connection explicitly. For fully NCG treatments to both integral and fractional QHE see e.g. [33–35].

The paper is organised as follows: in Section 2 we discuss Bellingeri's presentation of the Riemann surface braid group [20] and show that its simplest unitary representations can be constructed via unitary representations of the Weyl–Heisenberg