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On quantum symmetries of compact metric spaces

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ABSTRACT

An action of a compact quantum group on a compact metric space (X, d) is (D)-isometric if the distance function is preserved by a diagonal action on $X \times X$. In this study, we show that an isometric action in this sense has the following additional property: the corresponding action on the algebra of continuous functions on X by the convolution semigroup of probability measures on the quantum group contracts Lipschitz constants. In other words, it is isometric in another sense due to Li, Quaegebeur, and Sabbe, which partially answers a question posed by Goswami. We also introduce other possible notions of isometric quantum actions in terms of the Wasserstein p-distances between probability measures on X for $p \geq 1$, which are used extensively in optimal transportation. Indeed, all of these definitions of quantum isometry belong to a hierarchy of implications, where the two described above lie at the extreme ends of the hierarchy. We conjecture that they are all equivalent.

Introduction

Operator-algebraic compact quantum groups were introduced in [1] as non-commutative analogues of the C^*-algebra of continuous functions on a compact group. Subsequently, this field has developed explosively, thereby offering many interesting directions for research.

A popular method for constructing examples begins with some algebraic or geometric structure, before considering the "largest" compact quantum group that acts to preserve it. Examples of these structures include operator algebras [2–4], finite-dimensional Hilbert spaces (perhaps equipped with a bilinear form) [5], finite graphs [6], and Riemannian manifolds [7].

At the time, it was surprising when Wang discovered [3] that even the "half classical" situation when a quantum group acts on an ordinary space (in that case a finite one) can lead to interesting and non-trivial objects. The quantum automorphism group of a finite set introduced in Wang's study is truly quantum (i.e., not the function algebra on a finite group) provided that the set acted upon is not excessively small.

In the same spirit, quantum automorphism groups of finite metric spaces were constructed in [8]. Banica's notion of what it means for a quantum group action to be isometric can be adapted to possibly infinite compact metric spaces, as in [9], where Goswami studied the problem of whether or not a universal quantum isometric action exists in this infinite setting.

Thus, in the present study, we consider the actions of compact quantum groups on (classical) compact metric spaces.

A quantum group action on a compact metric space (X, d) is (D)-isometric if it satisfies the conditions from [9]. Roughly speaking, this requires that the distance function $d : X \times X \to \mathbb{R}_{\geq 0}$ is invariant under a certain "diagonal" action of the quantum group on $X \times X$ (see Section 1.3 for precise definitions).

A constant theme in non-commutative geometry is that the classical concepts that need to be generalized often lack one canonical "best" formulation in the quantum or non-commutative setting. As a consequence, classical notions sometimes

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bifurcate (or tri- or multi-furcate) into competing non-commutative analogues, which are not always obviously equivalent. The quantum symmetries of compact metric spaces are a case in point, where various studies have attempted to define what it means for a quantum action on a compact metric space (X, d) to be isometric. Another such proposal (in addition to that alluded to in the previous paragraph) was made in [10] (see Section 1.3 for a more detailed discussion).

By definition, a compact quantum group is a C^*-algebra A (see Definition 1.1 below) with some additional structure, so we can think of the states of A as probability measures on the underlying non-commutative space of the quantum group. Moreover, the comultiplication $A \to A \otimes A$ induces an associative binary operation on the set of states $S(A)$, thereby making it into a semigroup.

An action of the compact quantum group A on a compact space X induces a left action $\triangleright$ of $S(A)$ on the set $\mathcal{C}(X)$ of continuous functions on X, as well as a right action $\triangleleft$ of $S(A)$ on the set $\mathrm{Prob}(X)$ of probability measures on X.

If X is equipped with a metric d, we write $L(f)$ for the value of the Lipschitz seminorm of $f \in \mathcal{C}(X)$:

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We call $L(f)$ the Lipschitz constant of f, which can be infinity, but the subspace of Lipschitz functions (those with finite Lipschitz constant) is dense in $\mathcal{C}(X)$. We say that the action of the quantum group on X is (Lip)-isometric if for any $\varphi \in S(A)$ and any $f \in \mathcal{C}(X)$, the Lipschitz constant of $\varphi \triangleright f$ is not larger than that of f.

In [10, Theorem 3.5], it was shown that quantum group actions on finite metric spaces are (Lip)-isometric if and only if they are (D)-isometric. The question of whether or not this is true for general compact metric spaces was posed in [9]. We cannot answer this fully, but one of its implications is part of Theorem 3.1:

Proposition. *A (D)-isometric quantum action on a compact metric space is (Lip)-isometric.*

Furthermore, we can say the following.

Embed $X \subset \mathrm{Prob}(X)$ in the obvious manner by regarding the points as Dirac delta measures. The distance function d on X can be extended to a metric on $\mathrm{Prob}(X)$ in many ways to induce the weak * topology on probability measures. Some popular extensions of d with this property are the so-called Wasserstein metrics on $\mathrm{Prob}(X)$ (see Section 1.4), as folllows.

For $\mu, \nu \in \mathrm{Prob}(X)$ and $p \geq 1$, we define

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{X \times X} d(x, y)^p \, \mathrm{d}\pi \right)^{\frac{1}{p}},$$

where $\Pi(\mu, \nu)$ is the space of probability measures on $X \times X$ with marginals μ and ν on the two Cartesian factors X. Because this is (Lip)-isometric, it can be rephrased as follows: the action of any $\varphi \in S(A)$ on $\mathrm{Prob}(X)$ contracts the W_1 metric. Thus, we also say that it is '(Lip_1)-isometric.'

It is now clear that we need to define condition (Lip_p) for a quantum action on (X, d) by analogy with (Lip_1). If A is the C^*-algebra of the compact quantum group, we simply require that any $\varphi \in S(A)$ contracts the metric W_p on $\mathrm{Prob}(X)$.

The content of Corollary 3.2 can now be phrased as follows.

Theorem. *A quantum action on a compact metric space condition (Lip_p) is stronger with a larger $p \geq 1$, and condition (D) is stronger than all (Lip_p).*

As indicated above, universal objects are important sources of examples of compact quantum groups. We often look for the largest quantum group that acts faithfully (in some appropriate sense) on a certain object, but it is often not clear whether such quantum groups exist. In [3], for instance, it was shown that a finite-dimensional C^*-algebra has a quantum automorphism group if it is commutative, but not otherwise. By contrast, a finite-dimensional C^*-algebra equipped with a trace always has a quantum automorphism group.

Finite metric spaces always have quantum automorphism groups [8], as do those that are embeddable in some Euclidean space $\mathbb{R}^n$ [9]. However, it is unclear whether all compact metric spaces possess them.

In this study, we employ a somewhat weaker notion of universality, as given by [10]. One of the problems posed by [10] can be paraphrased as follows: given a faithful action of a quantum group on a compact metric space, is there a largest quantum subgroup that acts isometrically?

The question in [10] applied to (Lip)-isometric actions but it was not answered fully. In Theorem 4.4, we show that the answer is always affirmative in the (D)-isometric case.

We make a few conjectures related to this problem in Sections 3 and 4.

The remainder of this paper is organized as follows,

Section 1 provides the requisite background details, as well as some auxiliary results.

In Section 2, we prove a measure-theoretic version of Hall's marriage theorem on the existence of matchings in bipartite graphs, which generalizes Hall's and other combinatorial results of the same nature, and this is used later in the proof of Theorem 3.1.

Section 3 contains the proof of the main result, Theorem 3.1. In addition, as a consequence of the discussion in that section, in Proposition 3.10, we show that the underlying map $\rho : \mathcal{C}(X) \to \mathcal{C}(X) \otimes A$ of a (D)-isometric coaction is one-to-one.

Finally, in Section 4, we prove that every quantum group that acts faithfully on a compact metric space has a largest quantum subgroup that acts (D)-isometrically (Theorem 4.4).