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Dirac structures in vakonomic mechanics

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ABSTRACT

In this paper, we explore dynamics of the nonholonomic system called *vakonomic mechanics* in the context of Lagrange–Dirac dynamical systems using a Dirac structure and its associated Hamilton–Pontryagin variational principle. We first show the link between vakonomic mechanics and nonholonomic mechanics from the viewpoints of Dirac structures as well as Lagrangian submanifolds. Namely, we clarify that Lagrangian submanifold theory cannot represent nonholonomic mechanics properly, but vakonomic mechanics instead. Second, in order to represent vakonomic mechanics, we employ the space $TQ \times V^*$, where a *vakonomic Lagrangian* is defined from a given Lagrangian (possibly degenerate) subject to nonholonomic constraints. Then, we show how *implicit vakonomic Euler–Lagrange equations* can be formulated by the *Hamilton–Pontryagin variational principle* for the vakonomic Lagrangian on the extended Pontryagin bundle $(TQ \oplus T^*Q) \times V^*$. Associated with this variational principle, we establish a Dirac structure on $(TQ \oplus T^*Q) \times V^*$ in order to define an *intrinsic vakonomic Lagrange–Dirac system*. Furthermore, we also establish another construction for the vakonomic Lagrange–Dirac system using a Dirac structure on $T^*Q \times V^*$, where we introduce a *vakonomic Dirac differential*. Finally, we illustrate our theory of vakonomic Lagrange–Dirac systems by some examples such as the vakonomic skate and the vertical rolling coin.

1. Introduction

Some backgrounds. In conjunction with optimal control design, much effort has been concentrated upon exploring geometric structures and variational principles of constrained systems (see, for instance, [1–6]). The motion of such constrained systems may be subject to a nontrivial distribution on a configuration manifold. For the case in which the given distribution is integrable in the sense of Frobenius theorem, the constraint is called holonomic, otherwise nonholonomic. It is well known that equations of motion for Lagrangian systems with holonomic constraints can be formulated by Hamilton's variational principle by incorporating holonomic constraints into an original Lagrangian through Lagrange multipliers. On the other hand, Hamilton's variational principle does not yield correct equations of motion for mechanical systems with nonholonomic constraints, but induces different mechanics instead. The correct equations of motion for nonholonomic mechanics can be developed from the Lagrange–d'Alembert principle. In other words, there are two different mechanics associated with systems with nonholonomic constraints. The first one is based on the Lagrange–d'Alembert principle and the corresponding equations of motion are called *nonholonomic mechanics*. The second one is called *vakonomic mechanics*

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(*mechanics of* ***v****ariational* ***a****xiomatic* ***k****ind*), which is purely variational and was developed by Kozlov [7]; the name of vakonomic mechanics was coined by Arnold [2]. Needless to say, both approaches are essentially different from the other: interesting comparisons between both of them can be found in [8,9].

Nonholonomic mechanics has been studied from the viewpoints of Hamiltonian, Lagrangian as well as Poisson dynamics (see [10]). Indeed, nonholonomic mechanics has many applications to engineering, robotics, control of satellites, etc., since it seems to be appropriate to model the dynamical behavior of phenomena such as rolling rigid-body, etc. (see [11]). On the other hand, vakonomic mechanics appears in some problems of optimal control theory (related to sub-Riemannian geometry) [12,13], economic growth theory [14], motion of microorganisms at low Reynolds number [15], etc. A geometric unified approach was developed in [16].

In mechanics, one usually starts with a configuration manifold Q; Lagrangian mechanics deals with the tangent bundle TQ, while Hamiltonian mechanics with the cotangent bundle T^*Q. It is known that nonholonomic and vakonomic mechanics can be described on extended spaces because of the presence of Lagrange multipliers. An interesting geometric approach to Lagrangian vakonomic mechanics on $TQ \times \mathbb{R}^m$ may be found in [17], while an approach on $T(Q \times \mathbb{R}^m)$ may be found in [18]. In particular, since an extended Lagrangian on $TQ \times \mathbb{R}^m$ or $T(Q \times \mathbb{R}^m)$ is clearly *degenerate*, we have to explore its dynamics by using Dirac's theory of constraints (see [19]). Another interesting approach may be found in [9], where the authors depart from $TQ \oplus T^*Q$, and its submanifold $W_0 = \Delta_Q \times_Q T^*Q$, where $\Delta_Q \subset TQ$, in order to develop an intrinsic description of vakonomic mechanics.

As shown in [20], degenerate Lagrangian systems with nonholonomic constraints may be described, in general, by a set of implicit differential–algebraic equations, where a key point in the formulation of such implicit systems is to make use of the *Pontryagin bundle* $TQ \oplus T^*Q$, namely the fiber product (or Whitney) bundle $TQ \oplus T^*Q$. To the best of our knowledge, the Pontryagin bundle was first investigated in [21] to aid in the study of the degenerate Lagrangian systems, which is the case that we also treat in the present paper. The iterated tangent and cotangent spaces TT^*Q, T^*TQ, and T^*T^*Q and the relationships among these spaces were investigated by Tulczyjew [22] in conjunction with the generalized Legendre transform, where a symplectic diffeomorphism $\kappa_Q : TT^*Q \to T^*TQ$ plays an essential role in understanding Lagrangian systems in the context of Lagrangian submanifolds. The relation between these iterated spaces and the Pontryagin bundle was also discussed in [23]. Furthermore, Courant [24] investigated the iterated spaces TT^*Q, T^*TQ, and T^*T^*Q in conjunction with the tangent Dirac structures.

The notion of Dirac structures was developed by Courant and Weinstein [25], Dorfman [26] as a unified structure of pre-symplectic and Poisson structures, where the original aims of these authors were to formulate the dynamics of constrained systems, including constraints induced from degenerate Lagrangians, as in [19,27], where we recall that Dirac was concerned with degenerate Lagrangians, so that the image $P \subset T^*Q$ of the Legendre transformation, called the set of *primary constraints* in the language of Dirac, need not be the whole space. The canonical Dirac structures can be given by the graph of the bundle map associated with the canonical symplectic structure or the graph of the bundle map associated with the canonical Poisson structure on the cotangent bundle, and hence it naturally provides a geometric setting for Hamiltonian mechanics. It was already shown by Courant [28] that Hamiltonian systems can be formulated in the context of Dirac structures, however, its application to electric circuits and mechanical systems with nonholonomic constraints was studied in detail by van der Schaft and Maschke [29], where they called the associated Hamiltonian systems with Dirac structures *implicit Hamiltonian systems*. On the other hand, Yoshimura and Marsden [20] explored on the Lagrangian side to clarify the link between an induced Dirac structure on T^*Q and a degenerate Lagrangian system with nonholonomic constraints and they developed a notion of *implicit Lagrangian systems* as a Lagrangian analogue of implicit Hamiltonian systems. Moreover, the associated variational structure with implicit Lagrangian systems was investigated in [30], where it was shown that the Hamilton–Pontryagin principle provides the standard implicit Lagrangian system. Another recent development that may be relevant with the Dirac theory of constraints was explored by Cendra, Etchechouryb and Ferraro [31] by emphasizing the duality between the Poisson-algebraic and the geometric points of view, related to Dirac's and of Gotay and Nester's work; and by Grabowska and Grabowski [32] where the authors explored the Dirac setting regarding Lie algebroids.

Goals of the paper. The main purpose of this paper is to explore vakonomic mechanics, in the Lagrangian setting, both in the context of the Dirac structure and its associated variational principle called the Hamilton–Pontryagin principle. Another important point that we will clarify is the link between Dirac structures and Lagrangian submanifolds for the case of vakonomic mechanics. The organization of the paper is given as follows.

In Section 2, we will briefly introduce the geometric setting of the iterated tangent and cotangent bundles as well as the Pontryagin bundle. In Section 3, we will shortly review the Lagrangian submanifold theory for mechanics and then we will show that nonholonomic mechanics cannot be formulated on Lagrangian submanifolds, since the pullback of a symplectic two-form to the submanifold does not vanish. In Section 4 we will review Dirac structures in nonholonomic mechanics, by using the induced Dirac structure on the cotangent bundle and we will show how a degenerate Lagrangian system can be developed in the context of Dirac structures, together with the associated Lagrange–d'Alembert principle. In Section 5, we will consider the extended tangent bundle $TQ \times V^*$, where an extended Lagrangian $\mathfrak{L}$, called *vakonomic Lagrangian*, is defined in association with a given Lagrangian L on TQ and with nonholonomic constraints. Then we will show that the vakonomic dynamics on $(TQ \oplus T^*Q) \times V^*$ can be obtained by the Hamilton–Pontryagin principle for $\mathfrak{L}$, which yields the *implicit vakonomic Euler–Lagrange equations*. In parallel with this variational setting, taking advantage of the presymplectic structures constructed on $(TQ \oplus T^*Q) \times V^*$, we will illustrate how the vakonomic analogue of the Lagrange–Dirac systems can be intrinsically developed by making use of the Dirac structure on $(TQ \oplus T^*Q) \times V^*$. We shall also show another