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Nontrivial 1-parameter families of zero-curvature representations obtained via symmetry actions

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ABSTRACT

In this paper we consider the problem of constructing a 1-parameter family α_λ of zero-curvature representations of an equation $\mathcal{E}$, by means of classical symmetry actions on a given zero-curvature representation α. By using the cohomology defined by the horizontal gauge differential of α, we provide an infinitesimal criterion which permits to identify all infinitesimal classical symmetries of $\mathcal{E}$ whose flow could be used to embed α into a nontrivial 1-parameter family α_λ of zero-curvature representations of $\mathcal{E}$. The results of the paper are illustrated with some examples.

1. Introduction

Since the early applications of the inverse scattering method to the computation of soliton solutions of Korteweg–de Vries equation [1,2], the notion of zero-curvature representation (ZCR) has been widely investigated in the general study of nonlinear partial differential equations. This notion originates by the observation that some nonlinear partial differential equations (PDEs) can be interpreted as integrability conditions of an auxiliary linear system of PDEs, which indeed defines a ZCR [3,4]. In particular, it is typical for an integrable system of PDEs to admit a ZCR α_λ which depends on some real parameter λ, usually referred to as the *spectral parameter*. The presence of such a parameter is crucial not only for the determination of exact solutions, via the inverse scattering method [1,5] or the finite gap integration method [6], but also to guarantee other remarkable attributes of integrable equations like, for instance, the existence of infinite hierarchies of conservation laws and parametric Bäcklund transformations [7,3,8]. However, only nontrivial parameters (see Section 2) are suitable for such applications of 1-parameter families of ZCRs.

Hence the problem of deciding whether a parameter is trivial or not is particularly relevant in the theory of ZCRs. This problem has been already studied in the paper [9], by identifying a cohomological obstruction to removability and providing an effective method for the elimination of trivial parameters.

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However, while studying a differential equation, it is not unusual to know only a nonparametric ZCR [10,11,3,4]. Then, in such cases one is naturally faced with the embedding problem of a given nonparametric ZCR α into a nontrivial 1-parameter family α_λ of ZCRs. Hence, due to the importance of this problem, various attempts have been already made to provide any effective embedding method. Among these the symmetry method, first suggested in [11,8] and further developed in the papers [12,10,13], is particularly representative.

In its original formulation, the symmetry method allows one to embed a given ZCR α into a 1-parameter family of ZCRs α_λ of $\mathcal{E}$, via the action on α of a 1-parameter group A_λ of projectable point symmetries of $\mathcal{E}$. However, in general, a 1-parameter group A_λ may be not "good" in the sense that the induced embedding may result in a trivial 1-parameter family α_λ. Hence, to solve this problem, the authors of [10] suggest to compare the symmetry algebras of $\mathcal{E}$ and its covering, and conjecture that "good" symmetry groups A_λ can be identified by a mismatch of these algebras. However, that conjecture remains unproved.

In this paper we further develop the symmetry method, by taking into consideration the action of any kind of classical symmetry, and prove an infinitesimal criterion which is particularly effective in the identification of "good" infinitesimal classical symmetries, i.e., those symmetries which can be used to embed α into a nontrivial family α_λ of ZCRs of $\mathcal{E}$. According to that criterion we show that, relatively to α, one may distinguish classical infinitesimal symmetries of $\mathcal{E}$ into *gauge-like symmetries* and *non gauge-like symmetries*. The first type of symmetries form a Lie sub-algebra of the Lie algebra of symmetries of $\mathcal{E}$ and only produce trivial 1-parameter families of ZCRs. On the contrary, any 1-parameter family α_λ constructed with the flow of a non gauge-like symmetry is nontrivial.

We note that Marvan recently formulated in [14] an embedding method which is alternative to the symmetry method discussed in this paper. Both methods may be considered completely algorithmic, however the symmetry method is computationally more simple than Marvan's method, when a non gauge-like symmetry exists.

The paper is organized as follows. In Section 2, in order to make the results of the paper more accessible, we review the main facts from the geometric theory of differential equations used throughout the paper. In particular we review basic facts on symmetries of an equation $\mathcal{E}$ and define the notions of ZCR, gauge transformation and that of a nontrivial 1-parameter family of ZCRs. Section 3 is devoted to the application of symmetries of an equation $\mathcal{E}$ in the construction of a 1-parameter family of ZCRs of $\mathcal{E}$. Then in Section 4 we prove the main theorem which allows one to identify infinitesimal gauge-like symmetries as well as non gauge-like ones, for a given ZCR. According to this theorem, only infinitesimal symmetries which are non gauge-like, for a ZCR α, may be used to construct a nontrivial 1-parameter family α_λ. The results of the paper are illustrated with some examples in Section 5.

2. Preliminaries on jet spaces and ZCRs

The paper is based on the geometric theory of differential equations, hence we assume that the reader is familiar with this theory. However, in order to make the main results accessible to a wide range of readers, we collected here some notations and basic facts used throughout the paper. The interested reader should refer to [15,9,16–18] for further details.

Jet spaces and symmetries of differential equations

Let $\pi : E \to M$ be a fiber bundle, with $dim\, M = n$ and $dim\, E = n+m$. For any $k \in \mathbb{N}$ we denote by $J^k(\pi)$ the manifold of kth order jets of sections of π and by $\pi_k : J^k(\pi) \to M$ the *k-order jet bundle* of sections of π. Denoting by $\{x_1, \dots, x_n\}$ local coordinates on M and by $\{u^1, \dots, u^m\}$ local fiber coordinates of π, the induced *natural coordinates* on $J^k(\pi)$ will be $\{x_i, u^j_\sigma\}$, where $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ and $\sigma = (\sigma_1, \dots, \sigma_n)$ is a multi-index of order $|\sigma| = \sigma_1 + \cdots + \sigma_n$ such that $0 \le |\sigma| \le k$.

Then, for any $h > k$, we will denote by $\pi_{h,k} : J^h(\pi) \to J^k(\pi)$ the *natural projection* of $J^h(\pi)$ onto $J^k(\pi)$, and denote by $J^\infty(\pi)$ the *infinite jet space* of sections of π. By definition $J^\infty(\pi)$ is the inverse limit of the sequence of surjections $M \xleftarrow{\pi} J^0(\pi) \xleftarrow{\pi_{1,0}} \cdots \xleftarrow{\pi_{k,k-1}} J^k(\pi) \xleftarrow{\pi_{k+1,k}} \cdots$, hence it is not a finite dimensional manifold but one may introduce a differential calculus on it by making use of standard constructions of differential calculus over commutative algebras [18].

Throughout the paper, we will denote by $C^\infty(M)$ the algebra of smooth functions on M, by $\mathcal{F}_k(\pi)$ the algebra of *smooth functions on* $J^k(\pi)$ and by $\Lambda^*\left(J^k(\pi)\right)$ the exterior algebra of differential forms on $J^k(\pi)$. Analogously, on $J^\infty(\pi)$ the algebra of *smooth functions* will be denoted by $\mathcal{F}(\pi)$ and the exterior algebra of *differential forms* will be denoted by $\Lambda^*(\pi)$; in particular one has that $\mathcal{F}(\pi) = \Lambda^0(\pi)$. Any h-form over $J^\infty(\pi)$ can be identified with an h-form on some finite order jet space, hence the *exterior differential* d naturally extends to the exterior algebra $\Lambda^*(\pi)$ and defines the de Rham complex of $J^\infty(\pi)$.

We also recall that the $\mathcal{F}(\pi)$-module $\mathcal{D}(\pi)$ of *vector fields* on $J^\infty(\pi)$ is, by definition, the module of all derivations Z of $\mathcal{F}(\pi)$ with a *filtration degree* $l \in \mathbb{N}$, i.e., such that $Z(\mathcal{F}_k(\pi)) \subseteq \mathcal{F}_{k+l}(\pi),\ \forall k \in \mathbb{N}$. In coordinates, any vector field Z can be identified with a formal series $Z = \sum_i \alpha_i \partial_{x_i} + \sum_j \sum_{|\sigma|\ge 0} \beta^j_\sigma \partial_{u^j_\sigma}$, with $\alpha_i, \beta^j_\sigma \in \mathcal{F}(\pi)$.

The *Lie derivative* of functions, vector fields or forms on $J^\infty(\pi)$ is defined in a completely algebraic way. For instance, the Lie derivative of a function $G \in \mathcal{F}(\pi)$ along a vector field $Z \in \mathcal{D}(\pi)$ is $L_Z(G) := Z(G)$, and the Lie derivative of $Y \in \mathcal{D}(\pi)$ along Z is $L_Z Y := [Z, Y]$. Whereas, the Lie derivative of a form $\omega \in \Lambda^*(\pi)$ along Z is defined as $L_Z\omega := i_Z(d\omega) + d(i_Z\omega)$, where i_Z denotes the *inner product* operation $i_Z : \Lambda^h(\pi) \longrightarrow \Lambda^{h-1}(\pi)$.

The *Cartan distributions* on $J^k(\pi)$ and $J^\infty(\pi)$ will be denoted by $\mathcal{C}^k(\pi)$ and $\mathcal{C}(\pi)$, respectively. We recall that $\mathcal{C}(\pi)$ is locally the annihilator of the Pfaffian system $\{\theta^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma+1_i} dx_i : |\sigma| \ge 0,\ j = 1, \dots, m\}$. Dually $\mathcal{C}(\pi)$ is locally generated by the commuting vector fields $D_i := \partial_{x_i} + \sum_{|\rho|\ge 0} \sum_{j=1}^m u^j_{\rho+1_i} \partial_{u^j_\rho}$, which are called *total derivative operators*.