Journal of Geometry and Physics 94 (2015) 199–208

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Invariant solutions to the conformal Killing–Yano equation on Lie groups

A. Andrada, M.L. Barberis *, I.G. Dotti

FaMAF-CIEM, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

ARTICLE INFO

Article history:
Received 5 January 2015
Received in revised form 20 March 2015
Accepted 3 April 2015
Available online 16 April 2015

Keywords:
Conformal Killing–Yano 2-forms
Lie groups
Left invariant metrics

ABSTRACT

We search for invariant solutions of the conformal Killing–Yano equation on Lie groups equipped with left invariant Riemannian metrics, focusing on 2-forms. We show that when the Lie group is compact equipped with a bi-invariant metric or 2-step nilpotent, the only invariant solutions occur on the 3-dimensional sphere or on a Heisenberg group. We classify the 3-dimensional Lie groups with left invariant metrics carrying invariant conformal Killing–Yano 2-forms.

1. Introduction

The concept of conformal Killing–Yano p-forms (also known in the literature as twistor forms or conformal Killing forms) on Riemannian manifolds was introduced by Tachibana in [1] for $p = 2$ and later by Kashiwada in [2] for general p. Applications of these forms to theoretical physics were found related to quadratic first integrals of geodesic equations, symmetries of field equations, conserved quantities and separation of variables, among others (see, for instance, [3–7]). More recently, since the work of Moroianu, Semmelmann [8,9], a renewed interest in the subject arose among differential geometers (see, for instance, [10–16]).

Next we give the basic definitions and recall some well known properties of conformal Killing–Yano p-forms.

A p-form ω on an n-dimensional Riemannian manifold (M, g) is called *conformal Killing–Yano* (CKY for short) if it satisfies the following equation:

$$\nabla_X \omega = \frac{1}{p+1} \iota_X d\omega - \frac{1}{n-p+1} X^* \wedge d^*\omega, \quad X \in \mathfrak{X}(M) \tag{1}$$

where ∇ is the Levi-Civita connection, X^* is the 1-form dual to X and $d^* = (-1)^{n(p+1)+1} * d *$ is the co-differential. If, moreover, ω is co-closed, that is $d^*\omega = 0$, then it is called *Killing–Yano* (see [17]).

For $p = 1$, a 1-form ω is conformal Killing–Yano if and only if its dual vector field U is conformal, that is, $\mathcal{L}_U g = \varphi g$ for some $\varphi \in C^\infty(M)$. A 1-form ω is Killing–Yano if and only if its dual vector field is Killing.

We list next some properties:

- If ω is a CKY p-form on M, then $*\omega$ is a CKY $(n-p)$-form, where $*$ is the Hodge-star operator. In particular, $*$ interchanges closed and co-closed CKY forms (see [18,9]).

* Corresponding author.
E-mail addresses: andrada@famaf.unc.edu.ar (A. Andrada), barberis@famaf.unc.edu.ar (M.L. Barberis), idotti@famaf.unc.edu.ar (I.G. Dotti).

http://dx.doi.org/10.1016/j.geomphys.2015.04.004
0393-0440/

- Conformal invariance: If ω is a CKY p-form on (M, g) and $\tilde{g} := e^{2f} g$ is a conformally equivalent metric, then the form $\tilde{\omega} := e^{(p+1)f}\omega$ is a CKY p-form on $(M, \tilde{g})$ (see [6]).
- The space of CKY p-forms has finite dimension $\leq \binom{n+2}{p+1}$ with equality attained on the standard n-sphere (see [9]).

It was shown in [15] that every Killing–Yano p-form on a compact quaternionic Kähler manifold is automatically parallel ($p \geq 2$). In [10] the authors prove that a compact simply connected symmetric space carries a non-parallel Killing–Yano p-form if and only if it is isometric to a Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$. The case of conformal Killing–Yano p-forms on a compact Riemannian product was considered in [16], proving that such a form is a sum of forms of the following types: parallel forms, pull-back of Killing forms on the factors, and their Hodge duals.

A family of examples of manifolds carrying CKY 2-forms is given by spheres, nearly Kähler manifolds and Sasakian manifolds. In this article we will search for other examples, not necessarily compact. We restrict ourselves to the case of Lie groups with a left invariant metric. In Section 2 we consider Eq. (1) for $p = 2$ in the left invariant setting and we give an equivalent condition at the Lie algebra level (Proposition 2.5). We show that strong restrictions to the existence of left invariant KY and CKY 2-forms appear when we consider a compact Lie group with a bi-invariant metric or a 2-step nilpotent Lie group with any left invariant metric. Indeed, in Section 3 we prove that the only compact Lie group with a bi-invariant metric that admits non-coclosed CKY 2-forms is $SU(2)$, and in Section 4 we prove that the only 2-step nilpotent Lie groups carrying non-coclosed CKY 2-forms are the Heisenberg groups. In Section 5 we classify the 3-dimensional Lie groups with left invariant metrics carrying CKY 2-forms, obtaining families of globally defined metrics admitting CKY 2-forms on each of $\mathbb{R}^3$, $S^1 \times \mathbb{R}^2$ and S^3, and give the coordinate expression of the metrics. Moreover, we determine which of them give rise to Sasakian structures.

2. CKY tensors on Lie groups

Let G be an n-dimensional Lie group and let $\mathfrak{g}$ be the associated Lie algebra of all left invariant vector fields on G. If T_eG is the tangent space of G at e, the identity of G, the correspondence $X \to X_e := x$ from $\mathfrak{g} \to T_eG$ is a linear isomorphism. This isomorphism allows to define a Lie algebra structure on the tangent space T_eG setting, for $x, y \in T_eG$, $[x, y] = [X, Y]_e$ where X, Y are the left invariant vector fields defined by x, y, respectively.

A left invariant metric on G is a Riemannian metric such that L_a, the left multiplication by a, is an isometry for every $a \in G$. Every inner product on T_eG gives rise, by left translations, to a left invariant metric. Thus each n-dimensional Lie group possesses a $\frac{1}{2}n(n + 1)$-dimensional family of left invariant metrics.

A Lie group equipped with a left invariant metric is a homogeneous Riemannian manifold where many geometric invariants can be computed at the Lie algebra level. In particular, the Levi-Civita connection ∇ associated to a left invariant metric g, when applied to left invariant vector fields, is given by:

$$2\langle \nabla_x y, z\rangle = \langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle, \quad x, y, z \in \mathfrak{g}, \tag{2}$$

where $\langle\ ,\ \rangle$ is the inner product induced by g on $\mathfrak{g}$. Note that $\nabla g = 0$ implies that ∇_x is a skew-symmetric endomorphism of $\mathfrak{g}$ for any $x \in \mathfrak{g}$.

A left invariant p-form ω on G is a p-form such that $L_a^*\omega = \omega$ for all $a \in G$. We will consider left invariant p-forms ω on (G, g) satisfying (1). Since $\nabla\omega$, $d\omega$ and $d^*\omega$ are left invariant as well, we will study $\omega \in \Lambda^p\mathfrak{g}^*$ satisfying (1) for $x \in \mathfrak{g}$. We will call such a form a conformal Killing–Yano p-form on $\mathfrak{g}$.

For $p = 1$, we have the following result.

Proposition 2.1. *Let $\langle\cdot, \cdot\rangle$ be an inner product on a Lie algebra $\mathfrak{g}$. Then any CKY 1-form on $\mathfrak{g}$ is KY. Equivalently, if g is a left invariant metric on the Lie group G then any left invariant conformal vector field on G is a Killing vector field.*

Proof. For an arbitrary Riemannian manifold (M^n, g), a vector field ξ is conformal if $\mathcal{L}_\xi g = \varphi g$ with $\varphi \in C^\infty(M)$ given by $\varphi = -\frac{2}{n}d^*\eta$, where η is the 1-form dual to ξ. As a consequence, if g is a left invariant metric on G and ξ is left invariant, then φ is constant on G. Therefore, ξ satisfies the equation $\mathcal{L}_\xi\, g = c\, g$, $c \in \mathbb{R}$. Evaluating in $y, z \in \mathfrak{g}$ we obtain

$$-\langle [\xi, y], z\rangle - \langle y, [\xi, z]\rangle = c\langle y, z\rangle. \tag{3}$$

Setting $y = z = \xi$ in the above equation and assuming $\xi \neq 0$, we get $c = 0$, hence, ξ is a Killing vector field on (G, g). □

Corollary 2.2. *Let G be an n-dimensional Lie group with a left invariant metric g. Then any left invariant CKY $(n-1)$-form on (G, g) is closed.*

Proof. Let ω be a left invariant CKY $(n-1)$-form on (G, g), then $\eta := *\omega$ is a CKY 1-form, hence Proposition 2.1 implies that η is KY. Therefore, $d^*\eta = 0$, that is, $*d*\eta = 0$, which implies $d\omega = 0$, as claimed. □

From now on, we will consider left invariant CKY 2-forms on (G, g), where g is a left invariant metric. With respect to g, any left invariant 2-form ω on G gives rise to a skew-symmetric endomorphism T of the tangent bundle of G defined by $\omega(X, Y) = g(TX, Y)$, for any X, Y vector fields on G, which is also left invariant. We will still denote by T the corresponding endomorphism of $\mathfrak{g}$. When ω is a left invariant CKY 2-form on (G, g), the associated endomorphism T of $\mathfrak{g}$ will be called