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Notes on quantum weighted projective spaces and multidimensional teardrops

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ABSTRACT

It is shown that the coordinate algebra of the quantum $2n+1$-dimensional lens space $\mathcal{O}(L_q^{2n+1}(\prod_{i=0}^n m_i; m_0, \dots, m_n))$ is a principal $\mathbb{C}\mathbb{Z}$-comodule algebra or the coordinate algebra of a circle principal bundle over the weighted quantum projective space $\mathbb{WP}_q^n(m_0, \dots, m_n)$. Furthermore, the weighted $U(1)$-action or the $\mathbb{C}\mathbb{Z}$-coaction on the quantum odd dimensional sphere algebra $\mathcal{O}(S_q^{2n+1})$ that defines $\mathbb{WP}_q^n(1, m_1, \dots, m_n)$ is free or principal. Analogous results are proven for quantum real weighted projective spaces $\mathbb{RP}_q^{2n}(m_0, \dots, m_n)$. The K-groups of $\mathbb{WP}_q^n(1, \dots, 1, m)$ and $\mathbb{RP}_q^{2n}(1, \dots, 1, m)$ and the K_1-group of $L_q^{2n+1}(N; m_0, \dots, m_n)$ are computed.

1. Introduction

Quantum weighted projective spaces were introduced in [1] as fixed points of the weighted $U(1)$-actions on the coordinate algebra of the quantum odd dimensional sphere $\mathcal{O}(S_q^{2n+1})$. The way the circle group acts mimics the action on the classical sphere that leads to weighted projective spaces, prime examples of orbifolds. Classically these actions are not free unless all weights are mutually equal in which case one obtains the usual complex projective spaces. In the noncommutative case the situation is much more subtle, thereby more interesting. The studies in [1] concentrated on the case $n = 1$ and revealed that if the first of the two weights is equal to one, then a suitable finite cyclic group action on $\mathcal{O}(S_q^3)$ produces a lens space which in turn admits the $U(1)$-action which is free and has the quantum weighted projective space as fixed points. Subsequently, it has been shown in [2] that the cyclic group action that defines the lens space which is non-free classically is free in the quantum case. The combination of these observations leads one to conclude that, in a remarkable contrast to the classical situation (and in contradiction to an erroneous claim made in [1, Theorem 3.2]), the $U(1)$-action on $\mathcal{O}(S_q^3)$ that defines the quantum weighted projective space with the first weight equal to one is free. In the classical situation this space is the teardrop orbifold with a single singular point at the north pole (i.e. at the point with Cartesian co-ordinates $(0, 0, 1)$). Its coordinate algebra is determined by a polynomial with a multiple root, and the existence of this multiple root prevents one from differentiation and hence from constructing a unique tangent plane at the singular point. From the algebraic point of view, freeness of the action is controlled by a rational function with the denominator a polynomial in q, which has a root at $q = 1$. The introduction of non-commutativity in the form of a parameter $q \neq 1$, splits the multiple root 1 of the defining polynomial into separate roots (powers of q), thus allowing for differentiation, and also restores the freeness of the defining

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action, since the controlling rational function of q is well-defined in this case. Intuitively, one can visualize this process as blowing up a singular point into a sequence of spheres.

In further development, it has been shown in [3] that it is possible to consider a suitable lens space with a free $U(1)$-action which yields the quantum weighted projective lines with no restriction on weights as fixed points. The cyclic group action on $\mathcal{O}(S_q^3)$ that defines these lens spaces, however, is not free.

The most recent development is the analysis of quantum weighted projective spaces for all n, but with a restriction on the weights $m_0, \ldots, m_n$ which classically corresponds to projective varieties isomorphic to $\mathbb{CP}^n$ [4]. When this restriction is imposed one can find an explicit description of generators and D'Andrea and Landi construct principal circle bundles over quantum weighted projective spaces. Fixed points of cyclic group actions on $\mathcal{O}(S_q^{2n+1})$ serve as total spaces for these bundles. The first aim of the present note is to extend these results to quantum weighted projective spaces with no restriction on weights, and also to show that the defining circle action on $\mathcal{O}(S_q^{2n+1})$ is free provided $m_0 = 1$. The second aim is to take a closer look at multidimensional quantum teardrops, i.e. n-dimensional quantum weighted projective spaces with all bar the last weights being equal to one. Classically such projective varieties are not isomorphic to projective spaces unless $n = 1$. We describe generators of multidimensional quantum teardrops, outline their representation theory and calculate their K-groups.

2. Quantum freeness and strong gradings

In noncommutative geometry, the notion of a free quantum group action on a noncommutative space is encoded in the notion of a principal coaction of the corresponding Hopf algebra on the coordinate algebra of the quantum space; see e.g. [5]. If the quantum group is in fact an Abelian classical group, the principality is equivalent to the notion of *strong grading* [6].

Let G be a group. A *G-graded algebra* $\mathcal{A}$ decomposes into a direct sum of subspaces $\mathcal{A}_g$ labelled by $g \in G$ such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$, for all $g, h \in G$. In case $\mathcal{A}_g\mathcal{A}_h = \mathcal{A}_{gh}$, for all $g, h \in G$, $\mathcal{A}$ is said to be *strongly G-graded*. We will write $|a|_G$ for the G-degree of $a \in \mathcal{A}$. Also, $|\mathcal{A}|_G$ will denote the subalgebra of $\mathcal{A}$ consisting of all the invariant elements, i.e. all elements with degree equal to the neutral element of G. As explained in [6, Section A.I.3.2], $\mathcal{A}$ is strongly G-graded if and only if, for all generators g of G, there exist a finite number of elements $a_i, b_i \in \mathcal{A}$ such that

$$|a_i|_G = g^{-1}, \qquad |b_i|_G = g \quad \text{and} \quad \sum_i a_i b_i = 1. \tag{2.1}$$

Out of a given G-graded algebra $\mathcal{A}$, a group epimorphism $\pi : G \to H$, and a group monomorphism $\varphi : K \to G$ one can construct the following group graded algebras. First, π induces an H-grading on $\mathcal{A}$ by setting, for all $h \in H$,

$$\mathcal{A}_h := \bigoplus_{g\in\pi^{-1}(h)} \mathcal{A}_g.$$

Second, φ yields a K-graded algebra

$$\mathcal{A}^{(K)} := \bigoplus_{k\in K} \mathcal{A}_{\varphi(k)} \subseteq \mathcal{A},$$

i.e. $|a|_K = k$ if $|a|_G = \varphi(k)$. It is shown in [6, Section A.I.3.1.b] that the above H- and K-gradings are strong provided the initial G-grading is strong. A converse to this statement can be proven in case of Abelian groups. More precisely

Lemma 2.1 (*cf. Lemma A.1 in [7]*)**.** *Consider a short exact sequence of Abelian groups:*

$$0 \longrightarrow K \xrightarrow{\ \varphi\ } G \xrightarrow{\ \pi\ } H \longrightarrow 0. \tag{2.2}$$

A G-graded algebra $\mathcal{A}$ is strongly graded if and only if the induced H-grading on $\mathcal{A}$ and K-grading on $\mathcal{A}^{(K)}$ are strong.

Note that in the case of sequence (2.2), $|\mathcal{A}|_H = \mathcal{A}^{(K)}$ and $|\mathcal{A}|_G = |\mathcal{A}^{(K)}|_K$.

In this note we deal solely with algebras graded by Abelian groups which can be fitted into an exact sequence (2.2). Specifically, $K = G = \mathbb{Z}$, H is the cyclic group $\mathbb{Z}_N$, $\pi : \mathbb{Z} \to \mathbb{Z}_N$ is the canonical projection $m \mapsto m \bmod N$ and $\varphi : \mathbb{Z} \to \mathbb{Z}$ is given by $m \mapsto mN$. $\mathbb{Z}$-gradings correspond to the circle actions, while $\mathbb{Z}_N$-gradings correspond to actions by the cyclic group of order N.

3. Quantum weighted projective and lens spaces

The algebra $\mathcal{O}(S_q^{2n+1})$ of coordinate functions on the quantum sphere is the unital complex $*$-algebra with generators $z_0, z_1, \ldots, z_n$ subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j, \tag{3.1a}$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{j=i+1}^{n} z_j z_j^*, \qquad \sum_{j=0}^{n} z_j z_j^* = 1, \tag{3.1b}$$