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Flow of fractal fluid in pipes: Non-integer dimensional space approach

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A R T I C L E I N F O

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A B S T R A C T

Using a generalization of vector calculus for the case of non-integer dimensional space we consider a Poiseuille flow of an incompressible viscous fractal fluid in the pipe. Fractal fluid is described as a continuum in non-integer dimensional space. A generalization of the Navier–Stokes equations for non-integer dimensional space, its solution for steady flow of fractal fluid in a pipe and corresponding fractal fluid discharge are suggested.

1. Introduction

A cornerstone of fractal fluids is the non-integer dimension [1–3]. The mass of fractal fluid satisfies a power law relation $M \sim R^D$, where M is the mass of the ball region with radius R, and D is the mass dimension [4]. Fractal fluid can be described by four different approaches: (1) Using the methods of "Analysis on fractals" [5–10] it is possible to describe fractal media; (2) An application of fractional-differential continuum models suggested in [11,12], and then developed in [13–18], where so-called local fractional derivatives [19] are used; (3) Applying fractional-integral continuum models suggested in [4,20–23] (see also [24–32]), where integrations of non-integer orders and a notion of density of states [4] are used; (4) Fractal media can be described by using the theory of integration and differentiation for a non-integer dimensional space [33–35].

Let us note that main difference of the continuum models with non-integer dimensional spaces form the fractional continuum models suggested in [4,20–23] may be reduced to the following. (a) Arbitrariness in the choice of the numerical factor in the density of states is fixed by the equation of the volume of non-integer dimensional ball region. (b) In the fractional continuum models suggested in [4,20,21], the differentiations are integer orders whereas the integrations are non-integer orders. In the continuum models with non-integer dimensional spaces the integrations and differentiations are defined for the spaces with non-integer dimensions.

In this paper, we consider approach based on the non-integer dimensional space. The power law $M \sim R^D$ can be naturally derived by using the integrations in non-integer dimensional space [33], where the mass dimension of fractal fluid is connected with the dimension of this space. A vector calculus for non-integer dimensional space proposed in this paper allows us to use continuum models with non-integer dimensional spaces to describe for fractal fluids. This is due to the fact that although the non-integer dimension does not reflect completely the geometric properties of the fractal media, it nevertheless permits a number of important conclusions about the behavior of fractal structures. Therefore continuum models with non-integer dimensional spaces can be successfully used to describe fractal fluids.

Integration over non-integer dimensional spaces are actively used in the theory of critical phenomena and phase transitions in statistical physics [36,37], and in the dimensional regularization of ultraviolet divergences in quantum field theory [33,38,39]. The axioms for integrations in non-integer dimensional space are proposed in [34,40] and this type of integration is considered in the

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book by Collins [33] for rotationally covariant functions. In the paper [34] a mathematical basis of integration on non-integer dimensional space is given. Stillinger [34] suggested a generalization of the Laplace operator for non-integer dimensional spaces also. Using a product measure approach, the Stillinger's methods [34] has been generalized by Palmer and Stavrinou [35] for multiple variables case with different degrees of confinement in orthogonal directions. The scalar Laplace operators suggested by Stillinger [34] and Palmer, Stavrinou [35] for non-integer dimensional spaces, have successfully been used for effective descriptions in physics and mechanics. The Stillinger's form of Laplacian for the Schrödinger equation in non-integer dimensional space is used by He [41–43] to describe a measure of the anisotropy and confinement by the effective non-integer dimensions. Quantum mechanical models with non-integer (fractional) dimensional space have been discussed in [34,35,44–52]. Recent progress in non-integer dimensional space approach also includes description of the fractional diffusion processes in non-integer dimensional space in [53], and the electromagnetic fields in non-integer dimensional space in [54–61].

Unfortunately, [34,35] proposed only the second order differential operators for scalar fields in the form of the scalar Laplacian in the non-integer dimensional space. A generalization of the vector Laplacian [62] for the non-integer dimensional space is not suggested in [34,35]. The first order operators such as gradient, divergence, curl operators are not considered in [34,35] also. In the work [61] the gradient, divergence, and curl operators are suggested only as approximations of the square of the Laplace operator. Consideration only the scalar Laplacian in the non-integer dimensional space approach greatly restricts us in application of continuum models with non-integer dimensional spaces for fractal fluids and material. For example, we cannot use the Stillinger's form of Laplacian for vector field $\mathbf{v}(\mathbf{r},t)$ in hydrodynamics of fractal fluids, in fractal theory of elasticity and thermoelasticity, in electromagnetic theory of fractal media to describe processes in the framework non-integer dimensional space approach.

In this paper, we propose to use a vector calculus for non-integer dimensional space, and we define the first and second orders differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for non-integer dimensional space. In order to derive the vector differential operators in non-integer dimensional space we use the method of analytic continuation in dimension. For simplification we consider rotationally covariant scalar and vector functions that are independent of angles. It allows us to reduce differential equations in non-integer dimensional space to ordinary differential equations with respect to r. The proposed operators allow us to describe fractal media to describe processes in the framework of continuum models with non-integer dimensional spaces. In this paper we describe a Poiseuille flow of an incompressible viscous fractal fluid in the pipe. A generalization of the Navier–Stokes equation for non-integer dimensional space to describe for fractal fluid are suggested. A solution of this equation for steady flow of fractal fluid in a pipe and corresponding fractal fluid discharge are derived.

2. Fractal fluids

A basic characteristic of fractal fluids is the non-integer dimensions such as mass or "particle" dimensions [4]. For fractal fluids the number of particles $N_D(W)$ or mass $M_D(W)$ in any region $W \subset \mathbb{R}^3$ of this fluid increase more slowly than the 3-dimensional volume $V_3(W)$ of this region. For the ball region W with radius R in an isotropic fractal fluid, this property can be described by the relation between the number of particles $N_D(W)$ in the region W of fractal fluid, and the radius R in the form

$$N_D(W) = N_0 (R/R_0)^D, \quad R/R_0 \gg 1, \tag{1}$$

where R_0 is the characteristic size of fractal fluid such as a minimal scale of self-similarity of a considered fractal fluid. The number D is called the "particle" dimension. It is a measure of how the fluid particles fill the space. The parameter D does not depend on the shape of the region W. Therefore fractal fluids can be considered as fluid with non-integer "particle" or mass dimension.

If the fractal fluid consists of particles with identical masses m_0, then relation (1) gives

$$M_D(W) = M_0 (R/R_0)^D, \quad R/R_0 \gg 1, \tag{2}$$

where $M_0 = m_0 N_0$. In this case, the mass dimension coincides with the "particle" dimension.

As the basic mathematical tool for continuum models of fractal fluids, we propose to use the integration and differentiation in non-integer dimensional spaces. In Section 7, we will show that the power-law relation (2) for an isotropic fractal fluid can be naturally derived by using the integration over non-integer dimensional space, where the space dimension is equal to the mass dimension of fractal fluid.

In order to describe fractal fluid by continuum models with non-integer dimensional spaces, we use the concepts of density of states $c_3(D,\mathbf{r})$ that describes how closely packed permitted places (states) in the space $\mathbb{R}^3$, where the fractal fluid is distributed. The expression $dV_D(\mathbf{r}) = c_3(D,\mathbf{r})dV_3$ is equal to the number of permitted places (states) between V_3 and $V_3 + dV_3$ in $\mathbb{R}^3$. The notation $d^D\mathbf{r}$ also will be used instead of $dV_D(\mathbf{r})$. Note that density of states and distribution function are different concepts, and it is impossible to describe all properties of fractal fluids by the distribution function only.

For fractal fluids, we can use the equation

$$dN_D(W) = n(\mathbf{r})\, dV_D(\mathbf{r}), \tag{3}$$

where $n(\mathbf{r})$ is a concentration of particles that describes a distribution of number of particles on a set of permitted places (possible states). The density of states is chosen such that $dV_D(\mathbf{r}) = c_3(D,\mathbf{r})\, dV_3$ describes the number of permitted states in dV_3.

The form of the function $c_3(D,\mathbf{r})$ is defined by symmetries of considered problem and properties of the described fractal fluid. A general property of density of states for fractal fluids is a power-law type of these functions that reflects a scaling property (fractality) of the fractal fluid. To simplify our consideration in this paper we will consider only isotropic fractal fluids with density of states that