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Higher index focus–focus singularities in the Jaynes–Cummings–Gaudin model: Symplectic invariants and monodromy

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ABSTRACT

We study the symplectic geometry of the Jaynes–Cummings–Gaudin model with $n = 2m - 1$ spins. We show that there are focus–focus singularities of maximal Williamson type $(0, 0, m)$. We construct the linearized normal flows in the vicinity of such a point and show that soliton type solutions extend them globally on the critical torus. This allows us to compute the leading term in the Taylor expansion of the symplectic invariants and the monodromy associated to this singularity.

1. Introduction

The theory of integrable systems started with the work of Liouville [1] where he defined the general concept of integrable systems and their integration by "quadratures". Let $\mathcal{M}$ be a symplectic space of dimension $2n$, the system is Liouville integrable if we can define n Hamiltonians H_i in involution such that "generically" $dH_1 \wedge dH_2 \wedge \cdots \wedge dH_n$ has maximal rank $r = n$. The remarkable Arnold–Liouville theorem states that the space $\mathcal{M}$ is then fibered by invariant n-dimensional Lagrangian tori. This is a *semi global* result as it gives a global information on the fiber, which is a torus, and it is sufficient to assert that the motion is quasi periodic almost everywhere. This fibration however contains singular fibers i.e. fibers containing points where the rank r is not maximal, which play a very important role for the global properties of the system, see e.g. [2]. In this work, we will concentrate on special singularities where the rank $r = 0$. They correspond to equilibrium points, which can be stable or unstable, and have also a very rich physical content.

An equilibrium point $x^{(0)}$ is a simultaneous critical point for all the H_i, in other words, it is a point at which the differential of the moment map μ vanishes. Expanding the H_i around such a point, we get n Poisson commuting quadratic forms Q_i.

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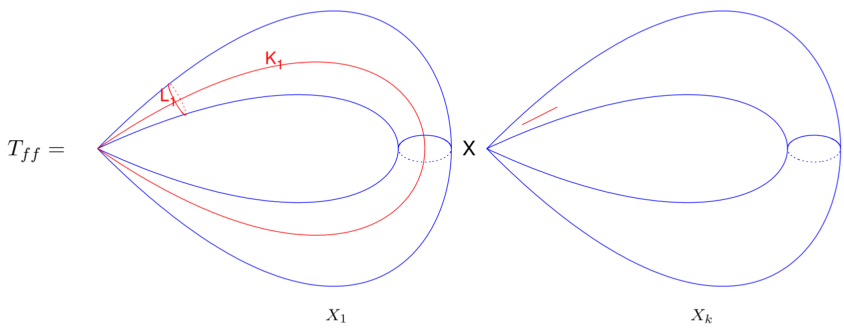

Fig. 1. The singular torus seen as a product of two dimensional pinched tori whose coordinates are the parameters X_k entering the solitonic solution of the equations of motion. A big motion (K_1) on one of the components of the torus induces a motion on the other components of the torus as well.

We shall consider the case of a purely focus–focus equilibrium point of Williamson type ($m_e = m_h = 0$ and $m_{ff} = m$). In this case, the dimension of $\mathcal{M}$ is equal to $4m$, and there exist canonical coordinates (p_i, q_i) such that the quadratic forms Q_i are linear combinations of the quadratic normal forms

$$K_j = p_{2j}q_{2j} + p_{2j+1}q_{2j+1}, \quad j = 1, \dots, m$$
$$L_j = -p_{2j}q_{2j+1} + p_{2j+1}q_{2j}, \quad j = 1, \dots, m.$$

By Eliasson's theorem [3], this description extends to a local neighborhood of the point x_0. There exists a *symplectic* diffeomorphism $\Phi : U \subset \mathbb{R}^{2n} \longrightarrow \mathcal{V} \subset \mathcal{M}$ mapping the neighborhood U of the origin to the neighborhood $\mathcal{V}$ of $x^{(0)}$, and a local diffeomorphism $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in a neighborhood of the origin such that:

$$\mu \circ \Phi = \psi \circ \mu^{(0)}$$

where $\mu^{(0)}$ is the quadratic moment map $(p, q) \to (K, L)$. An important consequence of this theorem is that it allows to extend the flows associated to the quadratic generators K_j and L_j from a neighborhood of $x^{(0)}$ (*local* level) to a family of fibers containing the level set of $x^{(0)}$ (*semi global* level). These flows will be called *normal flows* and will play an important role in this paper.

Using the complex variables

$$w_j = p_{2j} + ip_{2j+1}, \qquad z_j = q_{2j} + iq_{2j+1}, \quad j = 1, \dots, m \tag{1}$$

we have

$$w_j\bar{z}_j = K_j + iL_j$$

hence, the level set $\mathcal{S}$ of the equilibrium point is the image, by the diffeomorphism Φ, of the product of m two dimensional components, themselves the union of two complex planes intersecting transversely

$$\mathcal{C}_j : w_j = 0 \quad \text{or} \quad z_j = 0, \quad j = 1, \dots, m.$$

It is a $2m$-dimensional cone $\mathcal{C}$

$$\mathcal{C} = \prod_{j=1}^{m} \mathcal{C}_j.$$

We see that in a neighborhood of the equilibrium point, the phase-space fibration induced by the moment map is symplectically equivalent to the fibration of a direct product of m independent integrable four dimensional dynamical systems with a focus–focus singularity. An important question is to see whether such a simple description holds also at the semi-global level.

In the Jaynes–Cummings–Gaudin model that we will consider in this paper, the level set of the equilibrium point is a compact pinched torus of dimension $2m$. A result of Tien Zung [4] asserts that, at the topological level, this pinched torus is equivalent to a product of m two dimensional pinched tori. But this is not true at the symplectic level, and it is the purpose of this work to compute symplectic invariants defined in [5,6] preventing this simple product decomposition of the singular fiber (see Fig. 1).

To achieve this goal, we will combine two information. The first one is a very simple description of the normal modes around the singularity, which provide a complete *local* description of the system. The second one, of a *semi global* nature, is provided by the explicit solitonic solutions of the equations of motion on the singular fiber. It turns out that the two objects, normal modes and solitons, match perfectly : solitons are just global extensions of normal modes to the full singular fiber. The $2m$ real parameters describing the initial conditions of the solitonic solution may be seen as coordinates on the singular fiber which precisely realize its decomposition into a product of m two dimensional singular tori, thereby extending to the whole fiber the decomposition provided by the normal coordinates in the vicinity of the critical point.