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On the number of invisible directions for a smooth Riemannian metric

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ABSTRACT

In this note we give a construction of a C^∞-smooth Riemannian metric on $\mathbf{R}^n$ which is standard Euclidean outside a compact set K and such that it has $N = n(n+1)/2$ invisible directions, meaning that all geodesic lines passing through the set K in these directions remain the same straight lines on exit. For example in the plane our construction gives three invisible directions. This is in contrast with billiard type obstacles where a very sophisticated example due to A. Plakhov and V. Roshchina gives 2 invisible directions in the plane and 3 in the space.

We use reflection group of the root system A_n in order to make the directions of the roots invisible.

1. The problem of invisibility

Consider a C^∞-smooth Riemannian metric g on $\mathbf{R}^n$ which is supposed to be standard Euclidean outside a compact set K. Geodesics of the metric outside the set K are straight lines and are deformed somehow inside K. Following [1], we say that the obstacle K is invisible in the direction v if every geodesic in the direction v passing the obstacle remains the same straight line. This direction v is called the direction of invisibility in this case. It is important question how many invisible direction can exist for a non-flat C^∞-smooth Riemannian metric. It was proved in [2] based on [3] and generalizing previous results [4,5] that the invisibility in all directions implies that the metric is isometric to Euclidean one. This is the so called lens rigidity phenomena (see also [6] for further developments). It is a natural question to ask how large the set of invisible directions can be for a C^∞-smooth Riemannian metric. In particular can it be large, or infinite, or maybe even of positive measure?

In [1] this question is studied for an analogous model of elastic collisions with a compactly supported obstacle. A very tricky construction of two invisible directions in the plane and three in the space is given in [1].

On the other hand there are sophisticated examples of Riemannian metrics which are perfect lenses (see [7] for further references) so that all directions are invisible. In these examples Riemannian metric is either of *finite smoothness* at the points of gluing or has singularities—in full agreement with the lens rigidity phenomenon [2].

In this note we show that in C^∞-smooth case one can construct a Riemannian metric with $N = n(n+1)/2$ invisible directions in $\mathbf{R}^n$.

Theorem 1.1. *There exists a family of C^∞-smooth non-flat Riemannian metrics g on $\mathbf{R}^n$ which are Euclidean outside a compact set K and having $N = n(n+1)/2$ invisible directions.*

The idea is that N is the number of components of the metric tensor and also is the number of positive roots of the root system A_n. It is not clear if other reflection groups can be used in a similar manner.

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Remark 1. A similar problem for conformally flat metrics can be posed also. Our construction in this case gives only one invisible direction. Analogously, one can construct metrics in the diagonal form with n invisible directions.

Remark 2. It is worth mentioning that there exist smooth Finsler non-flat metrics which are compactly supported and have all the directions invisible. This is a very well known observation related to Hopf rigidity. These metrics can be constructed by the action of small compactly supported symplectic diffeomorphism of $T^*\mathbf{R}^n$ on the Lagrangian foliation corresponding to Minkowskii metric.

Remark 3. One can use this construction on other manifolds also. The most natural is to implant such a metric into a small ball of flat torus. Then the resulting geodesic flow has $N = n(n+1)/2$ invariant Lagrangian tori. It would be interesting to understand geometry and dynamics of these examples further.

2. Using generating functions

In this section we use generating functions to create Lagrangian submanifolds in the energy level of the Riemannian metric in question.

Throughout the paper we fix standard basis $e_1, \ldots, e_{n+1}$ and standard Euclidean scalar product $(\, , \,)$ on $\mathbf{R}^{n+1}$. Consider $\mathbf{R}^n \subset \mathbf{R}^{n+1}$, where $\mathbf{R}^n$ is viewed as hyperplane of $\mathbf{R}^{n+1}$ orthogonal to the vector $e_1 + \cdots + e_{n+1}$. We keep the same notation for the restriction of the scalar product $(\, , \,)$ to $\mathbf{R}^n$. Recall that the root system A_n can be realized as a set of the integer vectors $e_i - e_j$ (of the length $\sqrt{2}$) in $\mathbf{R}^n \subset \mathbf{R}^{n+1}$.

For our purposes it will be convenient to arrange the roots in the following order. Let $v_1, \ldots, v_N$ be such that first n are defined by $v_i = e_i - e_{n+1}$ and the rest are $e_i - e_j$ for $1 \le i < j \le n$. So there are $N = n(n+1)/2$ of them and together with their negatives they form all the roots. Notice that $(v_1, \ldots, v_n)$ form a basis of $\mathbf{R}^n$ and the rest are their differences $v_i - v_j$, $1 \le i < j \le n$.

For every $k = 1, \ldots, N$ consider the Lagrangian sections L_k of $T^*\mathbf{R}^n$ equipped with the standard symplectic structure which are defined by the generating function

$$S_k(x) = (v_k, x) + \epsilon\varphi_k(x), \qquad L_k = \{p = \nabla S_k = v_k + \epsilon\nabla\varphi_k\}, \tag{1}$$

where φ_k are any smooth functions on $\mathbf{R}^n$ with the support in a ball B. In this formula and later we shall identify vectors with co-vectors with the help of the scalar product $(\, , \,)$ and write ∇ for the standard Euclidean gradient.

It is a well known fact that any root system determines the scalar product uniquely. As a corollary of this we have the following

Theorem 2.1. *If $\epsilon > 0$ is small enough then there exists and unique Riemannian metric g on $\mathbf{R}^n$ such that all $L_k, \ k = 1, \ldots, N$ lie in the level $\{h = 1\}$ of the corresponding Hamiltonian function h. Moreover this metric coincides with the standard Euclidean $(\, , \,)$ outside the ball B.*

Proof. Write the Riemannian metric g and the Hamiltonian function h using coordinates $(x_1, \ldots, x_n)$ with respect to an orthonormal basis on $\mathbf{R}^n$:

$$g = \sum_{i,j=1}^{n} g_{ij}dx_idx_j = (Gdx, dx); \qquad h = \frac{1}{2}\sum_{i,j=1}^{n} h_{ij}p_ip_j = \frac{1}{2}(Hp, p),$$

where G, H are the matrices $G = (g_{ij}), \ H = (h_{ij}) = G^{-1}$.

For a given choice of the functions $\varphi_k, \ k = 1, \ldots, N$, the requirements that for every k, $L_k \subset \{h = 1\}$ form a linear system of N inhomogeneous equations on the N unknown coefficients h_{ij}:

$$\frac{1}{2}\sum_{i,j=1}^{n} h_{ij}\frac{\partial S_k}{\partial x_i}\frac{\partial S_k}{\partial x_j} = 1, \quad k = 1, \ldots, N. \tag{2}$$

Notice that for $\epsilon = 0$ this system reads:

$$\frac{1}{2}\sum_{i,j=1}^{n} h_{ij} \cdot (v_k)_i \cdot (v_k)_j = 1, \tag{3}$$

which simply means that all the vectors v_k have length $\sqrt{2}$. System (3) has unique solution namely the standard Euclidean metric, i.e., $G = H = \mathrm{Id}$ (since there exists and unique scalar product such that the length of every vector v_k equals $\sqrt{2}$). Therefore, the determinant of the system (3) is not zero and then for ϵ small enough there is a unique solution of (2) also which is a positive definite form. Moreover, outside the ball B (2) coincides with (3) and thus $G = H = \mathrm{id}$. $\square$

Remark 4. In principle one could find the solution h_{ij} of this linear system explicitly in terms of derivatives of the functions φ_k, thus determining the metric coefficients (see also Section 4).