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Topological monodromy as an obstruction to Hamiltonization of nonholonomic systems: Pro or contra?

A.V. Bolsinov [a,*], A.A. Kilin [b], A.O. Kazakov [c]

[a] Department of Mathematical Sciences, Loughborough University, LE11 3TU, UK
[b] Institute of Computer Science, ul. Universitetskaya 1, Izhevsk, 426034, Russia
[c] Lobachevsky State University of Nizhni Novgorod, 23 Prospekt Gagarina, 603950, Nizhni Novgorod, Russia

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ABSTRACT

The phenomenon of a topological monodromy in integrable Hamiltonian and nonholonomic systems is discussed. An efficient method for computing and visualizing the monodromy is developed. The comparative analysis of the topological monodromy is given for the rolling ellipsoid of revolution problem in two cases, namely, on a smooth and on a rough plane. The first of these systems is Hamiltonian, the second is nonholonomic. We show that, from the viewpoint of monodromy, there is no difference between the two systems, and thus disprove the conjecture by Cushman and Duistermaat stating that the topological monodromy gives a topological obstruction for Hamiltonization of the rolling ellipsoid of revolution on a rough plane.

1. Introduction

The paper has been motivated by the following general question in classical mechanics: how and to what extent does the dynamical behavior of nonholonomic systems differ from that of Hamiltonian ones? This question is closely related to the Hamiltonization problem: is it possible to turn a given nonholonomic system into Hamiltonian by an appropriate choice of a Poisson structure and change of time? This problem is quite nontrivial, discussed in many papers (see, e.g., [1–12]) and has many aspects, one of which is finding topological obstructions to Hamiltonization of integrable nonholonomic systems.

Here by integrability we understand the existence of sufficiently many first integrals such that their common regular levels are diffeomorphic to two-dimensional tori (as in the case of integrable Hamiltonian systems with two degrees of freedom). The phase space of such a system is foliated into invariant 2-tori. Speaking of topological obstructions to Hamiltonization, we mean the following natural question: is it possible to find those properties of such a foliation which allow us to distinguish it from similar foliations that appear in integrable Hamiltonian systems (the so-called Liouville foliations)?

Clearly, no such obstructions exist near a regular fiber. Moreover, it is well known that in the presence of an invariant measure the system (after an appropriate change of time) admits a Hamiltonian representation (see [13–15]). However, topological obstructions may exist in a neighborhood of singular fibers. One of such obstructions is the so-called topological monodromy of a foliation into invariant tori. The difference between Hamiltonian and non-Hamiltonian monodromy was one of the main issues studied in the famous paper by J. Duistermaat and R. Cushman [16] where a detailed topological treatment of the monodromy in integrable nonholonomic systems was given. As a concrete example of a nonholonomic

* Corresponding author. Tel.: +44 0 1509 222857; fax: +44 0 1509 223969.
E-mail addresses: A.Bolsinov@lboro.ac.uk (A.V. Bolsinov), aka@rcd.ru (A.A. Kilin), kazakovdz@yandex.ru (A.O. Kazakov).

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system, where the monodromy is essentially non-Hamiltonian and Hamiltonization is, therefore, impossible, the authors suggest the problem of the rolling prolate ellipsoid of revolution on a rough plane (i.e., rolling without sliding).[1]

However, it is well known that quite a similar problem in the case of a smooth plane (i.e., when the friction is zero) is Hamiltonian. Thus, it would be very interesting to observe any difference in the dynamics of these two systems. Since the monodromy is a rather rough topological characteristic, the phenomenon should be easy to observe. Our preliminary considerations, however, did not reveal any difference in the behavior of these systems and we decided to carry out a detailed analysis of the topological monodromy for both of them.

The paper is organized as follows. In the next section we recall the notion of monodromy for integrable systems and discuss some of its properties in the case of Hamiltonian systems. In particular, following [16], we make an emphasis on the difference between Hamiltonian and non-Hamiltonian cases. Then we discuss one of possible methods for calculating monodromy in systems with rotational symmetry, which is based on analysis of some properties of the Poincaré map for a specially chosen section. In Sections 3 and 4 we apply this method to study the monodromy in two integrable problems of a rolling prolate ellipsoid of revolution: on a smooth plane (Hamiltonian case) and on a rough plane (nonholonomic case).

The main conclusion of our work is that from the viewpoint of monodromy these two systems behave absolutely in the same way. In particular, the monodromy does not give any obstruction to Hamiltonization of this nonholonomic system. Moreover, our analysis shows, in fact, that the foliations into invariant tori in these two cases are isomorphic. However, this does not mean that the monodromy is useless for the Hamiltonization problem. On the contrary, it makes it possible to essentially reduce the "searching sector" for a suitable Poisson structure. These conclusions are discussed in the closing section of the paper.

2. Topological monodromy in integrable systems

The notion of a monodromy for integrable (Hamiltonian) systems was introduced by Duistermaat in [17] as one of obstructions to the existence of global action–angle variables. Since this notion has a pure topological nature, i.e. it is completely defined by the properties of the foliation into invariant tori, we can easily extend it to the case of nonholonomic integrable systems.

We recall the definition of monodromy in the case we are dealing with (some generalizations are discussed in [18, 19]). Consider an integrable system whose phase space is foliated into two-dimensional invariant submanifolds (tori). The singular fibers are ignored or just removed. Choose a particular torus T_0 and some deformation of it T_t, $t \in [0, 1]$, such that $T_1 = T_0$. In other words, we consider a closed path in the space of parameters (i.e., values of the first integrals) that defines a deformation after which the torus returns to the initial position.

Next we fix a pair of basis cycles λ_0, μ_0 on the initial torus T_0 and, by changing them continuously in the process of deformation, we obtain a family of cycles λ_t, μ_t forming a basis on T_t for each fixed value of $t \in [0, 1]$. When the deformation is completed, on the torus $T_1 = T_0$ we obtain a pair of basis cycles λ_1, μ_1. It is clear that if the deformation takes place inside a small neighborhood of T_0, then the cycles so obtained are homologous to the initial cycles λ_0, μ_0, i.e. λ_0 and λ_1 can be continuously deformed to each other inside T_0 (similarly for μ_0 and μ_1). However if the family T_t goes "far" from the initial torus T_0, it may happen that new cycles λ_1, μ_1 are essentially different from λ_0, μ_0. They nevertheless still form a basis and therefore, up to a homotopy, are related to the initial cycles by means of a certain integer unimodular matrix:

$$\begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \mu_0 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z},\ ad - bc = 1.$$

That is exactly what is called the *monodromy matrix* corresponding to the deformation T_t, $T_0 = T_1$. If it is different from the identity matrix we say that the monodromy is non-trivial.

Let us make some general comments about the monodromy which clarify its nature.

If we consider the foliation of the phase space $\mathcal{M}^4$ into invariant manifolds,[2] related to two integrals H and F, then it is convenient to consider the integral map $\Phi = (H, F)\colon \mathcal{M} \to \mathbb{R}^2$, its image $\Phi(\mathcal{M})$ and the bifurcation diagram $\Sigma \subset \Phi(\mathcal{M}) \subset \mathbb{R}^2$. Then choosing an initial torus T_0 is equivalent to choosing a non-singular (that is lying outside of Σ) point $a \in \Phi(\mathcal{M})$. The torus T_0 itself is the preimage of a. The deformation of the torus is defined by choosing a closed curve $\gamma(t)$ in the image of the integral map which does not intersect the bifurcation diagram (here we, or course, assume that $\gamma(0) = \gamma(1) = a$). The curve γ defines a deformation of the torus $T_t = \Phi^{-1}(\gamma(t))$ and, consequently, the monodromy.

If the curve γ in the image of the momentum map is continuously deformed in such a way that the deformation does not touch the bifurcation diagram, then the monodromy will not change. In particular, a non-trivial monodromy may appear for non-contractible loops γ only. Such non-contractible curves do not always exist, but very often they do, in particular, if

1 Here is a citation from [16]: "Because the monodromy going around this heteroclinic cycle is the identity, the rolling prolate ellipsoid of revolution cannot be made into a Hamiltonian system, even though it is time reversible and energy conserving. This is an example where a global invariant (namely, monodromy) has been used to show that a 4-dimensional conservative time reversible system is not Hamiltonian". Unfortunately, the paper does not contain any detailed explanations to this conclusion.

2 This construction does not change if we consider a dynamical system on a five-dimensional space $\mathcal{M}^5$ which admits three integrals H, F_1, F_2. The rolling ellipsoid on a plane is a system of this kind.