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Generic hyperelliptic Prym varieties in a generalized Hénon–Heiles system

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ABSTRACT

It is known that the Jacobian of an algebraic curve which is a 2-fold covering of a hyperelliptic curve ramified at two points contains a hyperelliptic Prym variety. Its explicit algebraic description is applied to some of the integrable Hénon–Heiles systems with a non-polynomial potential. Namely, we identify the generic complex invariant manifolds of the systems as a hyperelliptic Prym subvariety of the Jacobian of the spectral curve of the corresponding Lax representation.

The exact discretization of the system is described as a translation on the Prym variety.

1. Introduction

Many algebraic completely integrable systems possess matrix Lax representations whose spectral curves admit symmetries (in particular, involutions). The Jacobians of the curves contain Abelian (Prym) subvarieties whose open subsets are identified with the complex invariant manifolds of the systems.

In most cases the Prym subvariety itself is not a Jacobian variety, in particular, due to the fact that it is not principally polarized. In low dimensions the subvariety can be related to the Jacobian of an algebraic curve via an isogeny, and the latter curve also appears as the curve of separation of variables for the system (see [1,2], for example).

On the other hand, as we know from [3,4], when the spectral curve $\widetilde{C}$ admits an involution σ with two fixed points, and it covers a hyperelliptic curve, say C, the corresponding Prym variety becomes the Jacobian of (another) hyperelliptic curve C', and thus can be referred to as a *hyperelliptic Prym variety*. Such a situation occurs, in particular, in finite-dimensional reductions (stationary flows) of the Sawada–Kotera and the Kaup–Kupershmidt hierarchies of PDEs (see [5]) and the associated integrable dynamical systems, such as the Hénon–Heiles systems described in [6,7].

The special case where the two branch points of the covering (the images of the two fixed points of the involution σ) are also related to one another by the hyperelliptic involution on C was considered in detail by several authors, see e.g., [8–10]. In that case, the covering curve $\widetilde{C}$ is also hyperelliptic, and the equation of the second curve C' representing the Prym variety is derived in a straightforward way. The corresponding applications to integrable systems were considered in [11,10,12–14].

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For the general case, $\widetilde{C}$ is not hyperelliptic, and an algorithm for calculating the second curve C' was given only recently in [15]. In the present paper we apply a modification of the latter method to describe the complex invariant manifolds of certain integrable generalizations of the Hénon–Heiles system. To be precise, it is shown that the invariant manifolds of the cases (i) and (iii) of these systems are the 2-dimensional Prym subvariety of the Jacobian of a trigonal spectral curve $\widetilde{C}$, and the second curve C' is identified with the curve associated with the separation of variables found previously in [16,17]. Moreover, for an exact discretization (Bäcklund transformation) $\mathcal{B}$ of the above systems constructed in [18], we describe each branch of $\mathcal{B}$ as a translation on the Prym variety.

2. Double cover of a hyperelliptic curve with two branch points

Consider a hyperelliptic genus g curve C: $y^2 = f(x)$, where $f(x)$ is a polynomial of degree $2g+1$ with simple roots. Any 2-fold covering of C ramified at two finite points $P = (x_P, y_P)$, $Q = (x_Q, y_Q) \in C$ (*which are not related to each other by the hyperelliptic involution on* C) can be written in the form[2]

$$\widetilde{C} \,:\, z^2 = y + h(x), \qquad y^2 = f(x),$$

where $h(x)$ is a polynomial of degree $g+1$ such that

$$h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x)$$

with $\rho(x)$ being a polynomial[3] of degree g. Thus $\widetilde{C}$ admits the involution $\sigma \;:\; (x, y, z) \mapsto (x, y, -z)$, with fixed points $(x_P, y_P, 0)$, $(x_Q, y_Q, 0) \in \widetilde{C}$. Then the genus of $\widetilde{C}$ is $2g$ and the following was shown by D. Mumford and S. Dalaljan [3,4]:

(1) The Jacobian of $\widetilde{C}$ contains two g-dimensional Abelian subvarieties: $\mathrm{Jac}(C)$ and the Prym subvariety $\mathrm{Prym}(\widetilde{C}, \sigma)$. The former is invariant with respect to the involution σ extended to $\mathrm{Jac}(\widetilde{C})$, whereas the latter is anti-invariant.
(2) $\mathrm{Prym}(\widetilde{C}, \sigma)$ is a principally polarized Abelian variety and, moreover, is the Jacobian of a *hyperelliptic* curve C'.

It was further shown recently by A. Levin [15] that the second curve C' can be written explicitly as

$$w^2 = h(x) + Z, \qquad Z^2 = h^2(x) - f(x) \equiv (x - x_P)(x - x_Q)\rho^2(x), \tag{1}$$

which is equivalent to the plane curve

$$[w^2 - h(x)]^2 = h^2(x) - f(x) \implies w^4 - 2h(x)w^2 + f(x) = 0.$$

The latter can be transformed to a standard hyperelliptic form which is given in [15].

Note that when the polynomial $f(x)$ is of even degree $2g+2$ and g is odd ($g = 1, 3, 5, 7, \ldots$), the above formulas are still valid. However, when g is even, a different result holds, which is described as follows.

Theorem 1. (a) *In the case when the polynomial* $f(x)$ *has even degree* $2g+2$ *and* $g = 2, 4, 6, \ldots,$ *any covering* $\widetilde{C} \to C$ *ramified at* 2 *finite points* $P = (x_P, y_P)$, $Q = (x_Q, y_Q) \in C$ *can be written in the form*

$$\left\{ y^2 = f(x),\ z^2 = \frac{y + h(x)}{x - x_P} \text{ or, equivalently, } z^2 = \frac{y + h(x)}{x - x_Q} \right\}, \tag{2}$$

where, $h(x)$ *is of degree at most* $g+1$ *and such that*

$$h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x), \tag{3}$$

for some polynomial $\rho(x)$.

(b) *The corresponding* Prym *variety is isomorphic to the Jacobian of a second genus* g *hyperelliptic curve* C', *which can be written in the form*

$$\left\{ Y^2 = h^2(x) - f(x) = (x - x_P)(x - x_Q)\rho^2(x),\ w^2 = \frac{Y + h(x)}{x - x_P} \text{ or, equivalently, } w^2 = \frac{Y + h(x)}{x - x_Q} \right\}. \tag{4}$$

The latter is transformed to the standard hyperelliptic form $v^2 = P_{2g+2}(u)$, *where* P_{2g+2} *is a polynomial of degree* $2g+2$, *by the birational transformation*

$$x = \frac{x_Q u^2 - x_P}{u^2 - 1}, \qquad w = \left(\frac{x_Q - x_P}{u^2 - 1} \right)^{g/2} v, \tag{5}$$

with inverse

$$v = \frac{w}{(x - x_Q)^{g/2}}, \qquad u = \frac{(x - x_Q)w^2 - h(x)}{\rho(x)\,(x - x_Q)}.$$

2 Here and below we identify a curve with its regularization.
3 Here x_P or x_Q may or may not coincide with roots of $\rho(x)$.