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The Chaplygin case in dynamics of a rigid body in fluid is orbitally equivalent to the Euler case in rigid body dynamics and to the Jacobi problem about geodesics on the ellipsoid

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ABSTRACT

The main goal of this paper is to demonstrate how the theory of invariants for integrable Hamiltonian systems with two degrees of freedom created by A.T. Fomenko, H. Zieschang, and A.V. Bolsinov helps to establish Liouville and orbital equivalence of some classical integrable systems. Three such systems are treated in the article: the Euler case in rigid body dynamics, the Jacobi problem about geodesics on the ellipsoid and the Chaplygin case in dynamics of a rigid body in fluid. The first two systems were known to be Liouville and even topologically orbitally equivalent (Fomenko, Bolsinov). Now we show that the Chaplygin system is orbitally equivalent to the Euler and Jacobi systems.

1. Introduction

Integrable Hamiltonian systems occur in many problems of geometry, mechanics, and physics and have been of growing interest within the past years. For their study many methods have been developed, from classical works of Poincaré, Liouville, Moser up to modern treatments (e.g., bi-Hamiltonian approach). At the end of the past century the theory of invariants based on the topological approach was suggested by A.T. Fomenko, H. Zieschang, A.V. Bolsinov, and others for the study of integrable Hamiltonian systems with two degrees of freedom. This theory allows to investigate different qualitative properties of such systems and to conclude whether two systems are equivalent (in some sense) or not. Primarily, we mean the following three types of equivalence: Liouville equivalence [1–3], topological [4,5] and smooth [6] orbital equivalence. For each type of equivalence, a non-degenerate integrable system restricted to a 3-dimensional isoenergy surface is assigned with an invariant (molecule) which is a graph with some numerical marks. The main result of the theory in question can now be formulated in the following way: two integrable Hamiltonian systems considered on non-degenerate isoenergy 3-surfaces are equivalent (in one of the senses mentioned) if and only if the corresponding invariants are the same. We should underline that this approach enables one to establish the equivalence of different systems without writing out analytic formulas which seems to be quite a complicated task. Moreover, it allows us to conclude that two systems are not equivalent if their invariants differ. Possibility of establishing the fact of non-equivalence without using the theory of invariants seems doubtful.

Since its creation, the theory of topological invariants was applied to the study of many classical integrable systems, in particular, integrable cases in rigid body dynamics [7–12] and integrable geodesic flows [7,13–19]. This way, A.V. Bolsinov

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and A.T. Fomenko proved the topological orbital equivalence between two famous integrable systems: the Euler case in rigid body dynamics and the Jacobi problem about geodesics on the ellipsoid [20]. At the same time, the smooth orbital equivalence does not hold [21]. Moreover, as shown in [22], these two systems are not topologically conjugate (for any values of their parameters). In this paper we calculate topological invariants for the integrable Chaplygin case in dynamics of a rigid body in fluid and conclude that this classical system is also orbitally equivalent to the two mentioned above.

2. Main definitions and outline of the general theory

In this section we recall some basic definitions and main results of the theory of invariants for integrable Hamiltonian systems. Many notions of this theory will be discussed rather briefly. For details we address the readers to the papers [1–6] and the book [7].

Definition 1. A *Hamiltonian system* with n degrees of freedom is a triple (M^{2n}, ω, H) where M^{2n} is a $2n$-dimensional smooth manifold endowed with a symplectic form ω, and $H \in C^\infty(M^{2n})$ is a smooth function on M^{2n} called the *Hamiltonian function*. A *Hamiltonian vector field* is defined as $v = \operatorname{sgrad} H = \omega^{-1} dH$. We shall also denote a Hamiltonian system by (M^{2n}, v).

Theorem 1 (*Liouville*)**.** *Suppose that the smooth functions $f_1, \dots, f_n$ are the first integrals of the Hamiltonian system (M^{2n}, ω, H) satisfying the following conditions:*

(1) *$f_1, \dots, f_n$ are functionally independent, i.e. their gradients are linearly independent almost everywhere on M;*
(2) *$\{f_i, f_j\} = 0$ for $i, j = 1, \dots, n$, i.e. the integrals commute with respect to the Poisson bracket determined by the symplectic structure;*
(3) *the vector fields* $\operatorname{sgrad} f_1, \dots, \operatorname{sgrad} f_n$ *are complete, i.e. the natural parameter on their integral trajectories is defined on the whole real axis.*

Let $T_\xi = \{x \in M \mid f_i(x) = \xi_i,\ i = 1, \dots, n\}$ be a regular common level for the functions $f_1, \dots, f_n$ (regularity means that the differentials $df_1, \dots, df_n$ are linearly independent for all $x \in T_\xi$). Then

(a) *T_ξ is a smooth Lagrangian submanifold which in case it is connected and compact is diffeomorphic to the n-dimensional torus T^n (the Liouville torus);*
(b) *a neighbourhood U of the torus T^n is fiberwise diffeomorphic to the direct product $T^n \times D^n$, i.e. the pre-image of $T^n \times \{pt\}$ under this diffeomorphism is a Liouville torus;*
(c) *in the neighbourhood U there exists a coordinate system (the action–angle variables) $s_1, \dots, s_n, \varphi_1, \dots, \varphi_n$ where $s_1, \dots, s_n$ are coordinates on the disk D^n depending only on the integrals $f_1, \dots, f_n$ and $\varphi_1, \dots, \varphi_n$ are standard angle coordinates on the torus, such that*
    (i) $\omega = \sum_{i=1}^n d\varphi_i \wedge ds_i$,
    (ii) *the Hamiltonian vector field takes the form $\dot{s}_i = 0$, $\dot{\varphi}_i = q_i(s_1, \dots, s_n)$, $i = 1, \dots, n$, and therefore determines a rectilinear winding (rational or irrational) on each of the Liouville tori.*

This classical result suggests the following definitions.

Definition 2. A Hamiltonian system (M^{2n}, ω, H) is called *Liouville integrable* if there exists a set of smooth functions $f_1, \dots, f_n$ satisfying the conditions 1–3 of Theorem 1.

Definition 3. The decomposition of the manifold M^{2n} into connected components of common level surfaces of the integrals $f_1, \dots, f_n$ is called the *Liouville foliation* corresponding to the Liouville integrable Hamiltonian system (M^{2n}, ω, H).

The Liouville foliation consists of regular leaves (they are Liouville tori in the compact case) which fill M almost in the whole and singular ones filling a set of zero measure on M. Theorem 1 describes therefore the local structure of the Liouville foliation in the neighbourhood of a regular leaf. According to this theorem, the Liouville foliation is trivial in the neighbourhood of a Liouville torus. From now on we shall consider only systems with compact leaves.

Definition 4. The mapping $\mathcal{F} = f_1 \times \cdots \times f_n : M^{2n} \to \mathbb{R}^n$ is called the *momentum mapping*.

Let $K \subset M^{2n}$ be the set of all critical points of the momentum mapping (i.e. the set of all points $x \in M$ such that $\operatorname{rank} d\mathcal{F}(x) < n$).

Definition 5. The set $\Sigma = \mathcal{F}(K) \subset \mathbb{R}^n$ of all critical values of F is called the *bifurcation diagram* of the momentum mapping.

Obviously, the points from Σ are images of the singular leaves of the Liouville foliation under the momentum mapping, and the regular values of $\mathcal{F}$ (belonging to $\mathcal{F}(M) \setminus \Sigma$) are images of the regular leaves (Liouville tori).

Now we want to answer a natural question: what integrable Hamiltonian systems should be called equivalent? Several definitions are possible. Let $X_1 = (M_1^{2n}, \omega_1, H_1)$ and $X_2 = (M_2^{2n}, \omega_2, H_2)$ be two such systems.

Definition 6. The systems X_1 and X_2 are called *topologically* (resp. *smoothly*) *conjugate* if there exists a homeomorphism (resp. diffeomorphism) $\tau : M_1 \to M_2$ mapping the flow σ_1^t corresponding to X_1 into the flow σ_2^t corresponding to X_2: $\tau \circ \sigma_1^t = \sigma_2^t \circ \tau$. In other words, the systems X_1 and X_2 can be obtained from each other by a change of coordinates (continuous or smooth).