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Character and multiplicity formulas for compact Hamiltonian G-spaces

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Let $K \subset G$ be compact connected Lie groups with common maximal torus T. Let (M, ω) be a prequantisable compact connected symplectic manifold with a Hamiltonian G-action. Geometric quantisation gives a virtual representation of G; we give a formula for the character χ of this virtual representation as a quotient of virtual characters of K. When M is a generic coadjoint orbit our formula agrees with the Gross–Kostant–Ramond–Sternberg formula. We then derive a generalisation of the Guillemin–Prato multiplicity formula which, for λ a dominant integral weight of K, gives the multiplicity in χ of the irreducible representation of K of highest weight λ.

1. Introduction

Let (M, ω) be a compact connected symplectic manifold, and let G be a compact connected Lie group acting on M in a Hamiltonian fashion with moment map $\mu : M \to \mathfrak{g}^*$. Assume that the equivariant cohomology class $[\omega + \mu]$ is integral. Let $L \to M$ be a complex Hermitian line bundle and let ∇ be a Hermitian connection on L whose equivariant curvature form is the equivariant symplectic form $\omega + \mu$. Suppose that the G-action on M lifts to a G-action on L which preserves ∇. Let J be a G-equivariant almost complex structure on M. This almost complex structure gives us a totally complex distribution Δ in the complexified tangent bundle to M, such that $TM \otimes \mathbb{C} = \Delta \oplus \overline{\Delta}$. This gives us a splitting $\Lambda^k(TM \otimes \mathbb{C}) = \sum_{i+j=k} \Lambda^i(\overline{\Delta}) \otimes \Lambda^j(\Delta)$, and thus a bigrading on the space of L-valued differential forms. We will take J to be compatible with ω; that is, ω is a $(1, 1)$-form and $\omega(v, Jv) > 0$ for all $p \in M$ and $v \in T_pM$.

Let $D : \Omega^k(M; L) \to \Omega^{k+1}(M; L)$ be the operator $D(s \otimes \alpha) = \nabla s \otimes \alpha + s \otimes d\alpha$, where $s \in \Gamma(L)$ and $\alpha \in \Omega^k(M)$. Define $\overline{\partial}$ to be the $(0, k + 1)$ component of D. Fix a Hermitian metric on M. Together with the Hermitian inner product on L, this gives a Hermitian inner product on $L \otimes \Lambda^{0,k}TM$. Define operators $\overline{\partial}^t : \Omega^{0,k}(M; L) \to \Omega^{0,k-1}(M; L)$ which are $\mathcal{L}^2$-adjoint to $\overline{\partial}$. Then the operator $\overline{\partial} + \overline{\partial}^t : \Omega^{0,\text{even}}(M; L) \to \Omega^{0,\text{odd}}(M; L)$ is elliptic; the *quantisation* $Q(M, \omega, \nabla, L; J)$ is defined as the virtual G-representation on $\ker(\overline{\partial} + \overline{\partial}^t) \ominus \operatorname{coker}(\overline{\partial} + \overline{\partial}^t)$; that is, the equivariant index of the $\overline{\partial} + \overline{\partial}^t$ operator on L. (See Chapter 6 of Ginzburg, Guillemin and Karshon's book [1].)

In their paper [2], Guillemin and Prato prove a formula for the multiplicity with which each irreducible character of G appears in the character of this representation $Q(M, \omega, L, \nabla; J)$. In the special case of a torus T acting on a coadjoint orbit G/T by left multiplication, their formula becomes the Kostant multiplicity formula that Kostant obtained in [3].

Let G be semisimple, let $K \subset G$ be a Lie subgroup of equal rank, choose a common maximal torus $T \subset K \subset G$, let λ be a dominant integral weight for G, and let M be the coadjoint orbit $G \cdot \lambda$. The choice of positive roots for G determines a

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complex structure on $G\cdot\lambda$; take J to be the corresponding almost complex structure on $G\cdot\lambda$. Let $L_\lambda = G\times_T \mathbb{C}_{(\lambda)}$ (where T acts on the line $\mathbb{C}_{(\lambda)}$ with weight λ). Then the quantisation $Q(M,\omega,L_\lambda,\nabla;J)$ is a G-representation on the space of holomorphic sections of L_λ (see [1] for details). The Borel–Weil theorem tells us that the quantisation of $(M,\omega,L_\lambda,\nabla,J)$ is the irreducible representation of G of highest weight λ, and that all irreducible representations arise in this way, as described by Bott in [4]. In this case, Gross, Kostant, Ramond and Sternberg provided in [5] a formula for the character of this G-representation as a quotient of the alternating sum of a multiplet of K-characters. Their formula has its origins in String Theory and is the motivation for our work which provides a generalisation. In the special case where $K = T$, their formula becomes the Weyl character formula.

In this paper, we extend the result of Gross, Kostant, Ramond and Sternberg by replacing the coadjoint orbit $G\cdot\lambda$ with any compact connected symplectic Hamiltonian G-manifold M, and relate the resulting character formula to the Guillemin–Prato multiplicity formula. In Section 2 we obtain, for arbitrary compact connected symplectic manifolds M with Hamiltonian G-actions, a formula for the character of $Q(M,\omega,\nabla,L;J)$ as a quotient of K-characters. In Section 3, we derive from our formula a generalisation of the Guillemin–Prato multiplicity formula.

2. Character formula

Let G and K be compact connected Lie groups of equal rank with $K\subset G$, and choose a common maximal torus $T\subset K\subset G$. Write $\mathfrak{t}$, $\mathfrak{k}$ and $\mathfrak{g}$ for the Lie algebras of T, K, and G respectively. Let $\mathcal{N}_G(T)$ denote the normaliser in G of T and let $W(G)=\mathcal{N}_G(T)/T$ be the Weyl group of G. Choose a set $\Phi^+(G)$ of positive roots of G, and let $\mathcal{W}_G\subset\mathfrak{t}^*$ be the positive Weyl chamber for G. Let $\Lambda\subset\mathfrak{t}^*$ denote the weight lattice. For $\phi\in\Phi^+(G)$, let H_ϕ denote the hyperplane orthogonal to ϕ in $\mathfrak{t}^*$, and let $w_\phi\in W(G)$ be the reflection in this hyperplane. Let (M,ω) be a compact connected symplectic manifold, and let G act on (M,ω) in a Hamiltonian manner. Then T acts on (M,ω) in a Hamiltonian fashion with T-equivariant moment map $\mu: M\to\mathfrak{t}^*$; we will assume that the fixed points of this torus action are isolated. Suppose the equivariant cohomology class $[\omega+\mu]$ is integral, choose a prequantisation line bundle (L,∇), and let J be a G-equivariant almost complex structure on M that is compatible with ω. Let $Q(M,\omega,\nabla,L;J)=\operatorname{ind}(\overline{\partial}+\overline{\partial}^t)$ be the quantisation of $(M,\omega,L,\nabla;J)$, and let χ denote its character. In this section we will give an expression for the character χ, as a sum of quotients of (virtual) K-characters. We begin by setting up the equivariant cohomology we need, and by recalling the equivariant index theorem and the localisation theorem which will be our main tools.

2.1. Review of equivariant cohomology

Let G be a compact Lie group acting on a manifold M. Let EG be a contractible space on which G acts freely, so that $M\times EG\simeq M$ and the diagonal action of G on $M\times EG$ is free. Form the homotopy quotient $M_G := (M\times EG)/G$.

Definition 2.1. The *equivariant cohomology ring* $H^*_G(M)$ is the ordinary cohomology ring $H^*(M_G)$.

Let $H\subset G$ be a subgroup. Then H also acts freely on EG, so we can take $EH=EG$ and thus $M_G=M_H/G$. If $p: M_H\to M_G$ denotes the projection, we can pull back classes in $H^*_G(M)$ along p to $H^*_H(M)$.

An alternative approach to defining equivariant cohomology is known as the Cartan model. Define an *equivariant differential form* to be a G-equivariant polynomial on $\mathfrak{g}$ taking values in $\Omega^*(M)$. More precisely, the *equivariant differential k-forms* are elements of $\Omega^k_G(M)=\bigoplus_{k=2i+j}(S^i(\mathfrak{g}^*)\otimes\Omega^j(M))^G$. The *equivariant exterior differential* $d_G:\Omega^k_G(M)\to\Omega^{k+1}_G(M)$ is given by

$$(d_G\alpha)(X)=d(\alpha(X))-i_{X_M}\alpha(X),$$

where $\alpha\in\Omega^k_G(M)$, $X\in\mathfrak{g}$, and X_M is the vector field defined by the infinitesimal action of X on M. Note that $d_G^2=0$ by invariance.

Hence, an equivariant form $\alpha\in\Omega^*_G(M)$ is closed if $d(\alpha(X))-i_{X_M}\alpha(X)=0$ for all $X\in\mathfrak{g}$.

Example 2.2 (*Equivariant Symplectic Form*)**.** Let $\omega+\mu$ be the equivariant symplectic form. Then $d_G(\omega+\mu)(X)=d(\omega+\mu)(X)-i_{X_M}(\omega+\mu)(X)$. But $d\omega=0$ since ω is the non-equivariant symplectic form, and $i_{X_M}(\mu)=0$ since μ is a 0-form. So $d_G(\omega+\mu)(X)=d\mu(X)-i_{X_M}\omega(X)$, which is zero by definition of the moment map μ. So the equivariant symplectic form is equivariantly closed.

Theorem 2.3 (*Equivariant de Rham Theorem*)**.** *Let G be a compact connected Lie group acting on a manifold M. Then the equivariant cohomology is given by*

$$H^*_G(M)=\frac{\ker d_G}{\operatorname{im} d_G}.$$

Suppose $S\subset T$ is a subtorus with Lie algebra $\mathfrak{s}$. A T-equivariant form α is an $\Omega(M)$-valued polynomial on $\mathfrak{t}$; by restriction we can view this as an $\Omega(M)$-valued polynomial on $\mathfrak{s}$ and hence as an S-equivariant form. If α is T-equivariantly closed then $d(\alpha(X))-i_{X_M}\alpha(X)=0$ for all $X\in\mathfrak{t}$, in which case we certainly have $d(\alpha(X))-i_{X_M}\alpha(X)=0$ for all $X\in\mathfrak{s}$ and so α is also S-equivariantly closed.