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Contact flows and integrable systems

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ABSTRACT

We consider Hamiltonian systems restricted to the hypersurfaces of contact type and obtain a partial version of the Arnold–Liouville theorem: the system need not be integrable on the whole phase space, while the invariant hypersurface is foliated on an invariant Lagrangian tori. In the second part of the paper we consider contact systems with constraints. As an example, the Reeb flows on Brieskorn manifolds are considered.

1. Introduction

Usually, integrable systems are considered within a framework of symplectic or Poisson geometry, but there is a well defined non-Hamiltonian (e.g., see [1,2]) as well as a contact setting studied in [3–8]. The aim of this paper is to stress some natural applications of contact integrability to the Hamiltonian systems and to provide examples of contact integrable flows.

In the first part of the paper we consider Hamiltonian systems restricted to the hypersurfaces of contact type and obtain a partial version of the Arnold–Liouville theorem: the system need not be integrable on the whole phase space, while the invariant hypersurface is foliated on invariant Lagrangian tori with quasi-periodic dynamics. The construction can also be applied to the partially integrable systems and the systems of Hess–Appel'rot type, where the invariant hypersurface is foliated on Lagrangian tori, but not with quasi-periodic dynamics [9–11].

In the second part of the paper we consider contact systems with constraints and derive a contact version of Dirac's construction for constrained Hamiltonian systems. As an example, the Reeb flows on Brieskorn manifolds are considered. From the point of view of integrability, those Reeb flows are very simple. However, we use them to clearly demonstrate the use of Dirac's construction and to interpret the regularity of the Reeb flows within a framework of contact integrability.

For the completeness and clarity of the exposition, in Sections 1.1, 1.2 and 2.1 we recall basic definitions in contact geometry, the notion of contact noncommutative integrability, and the Maupertuis–Jacobi metric, respectively.

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1.1. Contact flows and the Jacobi bracket

A *contact form* α on a $(2n+1)$-dimensional manifold M is a Pfaffian form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. By a *contact manifold* $(M, \mathcal{H})$ we mean a connected $(2n+1)$-dimensional manifold M equipped with a nonintegrable *contact* (or *horizontal*) *distribution* $\mathcal{H}$, locally defined by a contact form: $\mathcal{H}|_U = \ker\alpha|_U$, U is an open set in M [12]. Two contact forms α and α' define the same contact distribution $\mathcal{H}$ on U if and only if $\alpha' = a\alpha$ for some nowhere vanishing function a on U.

The condition $\alpha \wedge (d\alpha)^n \neq 0$ implies that the form $d\alpha|_x$ is nondegenerate (symplectic) structure restricted to $\mathcal{H}_x$. The conformal class of $d\alpha|_x$ is invariant under the change $\alpha' = a\alpha$. If $\mathcal{V}$ is a linear subspace of $\mathcal{H}_x$, then we have well defined its orthogonal complement in $\mathcal{H}_x$ with respect to $d\alpha|_x$, as well as the notion of the *isotropic* ($\mathcal{V} \subset \operatorname{orth}_{\mathcal{H}} \mathcal{V}$), *coisotropic* ($\mathcal{V} \supset \operatorname{orth}_{\mathcal{H}} \mathcal{V}$) and the *Lagrange* subspaces ($\mathcal{V} = \operatorname{orth}_{\mathcal{H}} \mathcal{V}$) of $\mathcal{H}_x$. A submanifold $N \subset M$ is *pre-isotropic* if it transversal to $\mathcal{H}$ and if $\mathcal{G}_x = T_xN \cap \mathcal{H}_x$ is an isotropic subspace of $\mathcal{H}_x$, $x \in N$. The last condition is equivalent to the condition that distribution $\mathcal{G} = \bigcup_x G_x$ defines a foliation. It is *pre-Legendrian* submanifold if it is of maximal dimension $n+1$, that is $\mathcal{G}$ is a Lagrangian subbundle of $\mathcal{H}$.

A *contact diffeomorphism* between contact manifolds $(M, \mathcal{H})$ and $(M', \mathcal{H}')$ is a diffeomorphism $\phi : M \to M'$ such that $\phi_*\mathcal{H} = \mathcal{H}'$. An *infinitesimal automorphism* of a contact structure $(M, \mathcal{H})$ is a vector field X, called a *contact vector field* such that its local 1-parameter group is made of contact diffeomorphisms. Locally, if $\mathcal{H} = \ker\alpha$, then $\mathcal{L}_X\alpha = \lambda\alpha$, for some smooth function λ.

From now on, we consider *co-oriented (or strictly) contact manifolds* (M, α), where contact distributions $\mathcal{H}$ are associated to globally defined contact forms α. The *Reeb vector field* Z is a vector field uniquely defined by

$$i_Z\alpha = 1, \qquad i_Z d\alpha = 0. \tag{1.1}$$

The tangent bundle TM and the cotangent bundle T^*M are decomposed into $TM = \mathcal{Z} \oplus \mathcal{H}$ and $T^*M = \mathcal{H}^0 \oplus \mathcal{Z}^0$, where $\mathcal{Z} = \mathbb{R}Z$ is the kernel of $d\alpha$, $\mathcal{Z}^0$ and $\mathcal{H}^0 = \mathbb{R}\alpha$ are the annihilators of $\mathcal{Z}$ and $\mathcal{H}$, respectively. The sections of $\mathcal{Z}^0$ are called *semi-basic forms*. Whence, we have decompositions of vector fields and 1-forms

$$X = (i_X\alpha)Z + \hat{X}, \qquad \eta = (i_Z\eta)\alpha + \hat{\eta}, \tag{1.2}$$

where $\hat{X}$ is horizontal and $\hat{\eta}$ is semi-basic.

The mapping $\alpha^\flat : X \mapsto -i_X d\alpha$ carries X into a semi-basic form. The form $d\alpha$ is non-degenerate on $\mathcal{H}$ and the restriction of $\alpha^\flat$ to horizontal vector fields is an isomorphism whose inverse will be denoted by $\alpha^\sharp$. The vector field

$$Y_f = fZ + \hat{Y}_f, \qquad \hat{Y}_f = \alpha^\sharp(\widehat{df}). \tag{1.3}$$

is a contact vector field ($\mathcal{L}_{Y_f}\alpha = Z(f)\alpha$) and

$$\dot{x} = Y_f \tag{1.4}$$

is called the *contact Hamiltonian equation* corresponding to f. Notice that $Z = Y_1$ and $f = i_{Y_f}\alpha$.

The mapping $f \mapsto Y_f$ is a Lie algebra isomorphism ($Y_{[f,g]} = [Y_f, Y_g]$) between the set $C^\infty(M)$ of smooth functions on M and the set $\mathcal{N}$ of infinitesimal contact automorphisms. Here, the *Jacobi bracket* $[\cdot,\cdot]$ on $C^\infty(M)$ is defined by (see [12])

$$[f, g] = d\alpha(Y_f, Y_g) + fZ(g) - gZ(f) = Y_f(g) - gZ(f). \tag{1.5}$$

The Jacobi bracket does not satisfy the Leibniz rule. However, within the class of functions that are integrals of the Reeb flow, $C^\infty_\alpha(M) = \{f \in C^\infty(M) \mid Z(f) = [1, f] = 0\}$, the Jacobi bracket has the usual properties of the Poisson bracket: g is an integral of (1.4) if and only if $[g, f] = 0$ and if g_1 and g_2 are two integrals of (1.4), then $[g_1, g_2]$ is also an integral.

1.2. Contact integrability

A Hamiltonian system on $2n$-dimensional symplectic manifold (P, ω) is *noncommutatively integrable* if it has $2n-r$ almost everywhere independent integrals $F_1, F_2, \dots, F_{2n-r}$ and $F_1, \dots, F_r$ commute with all integrals

$$\{F_i, F_j\} = 0, \quad j = 1, \dots, r, \; i = 1, \dots, 2n - r. \tag{1.6}$$

Regular compact connected invariant manifolds of the system are isotropic tori, generated by Hamiltonian flows of $F_1, \dots, F_r$. In a neighborhood of a regular torus, there exist canonical *generalized action–angle coordinates* such that integrals F_i, $i = 1, \dots, r$ depend only on actions and the flow is translation in angle coordinates (see Nekhoroshev [13] and Mishchenko and Fomenko [14]). If $r = n$ we have the usual commutative integrability described in the Arnold–Liouville theorem [15].

Noncommutative integrability of contact flows can be stated in the following form (see [7]). Suppose we have a collection of integrals $f_1, f_2, \dots, f_{2n-r}$ of Eq. (1.4) on a $(2n+1)$-dimensional co-oriented contact manifold (M, α), where $f = f_1$ or $f = 1$ and

$$[1, f_i] = 0, \quad [f_i, f_j] = 0, \quad i = 1, \dots, 2n - r, \; j = 1, \dots, r. \tag{1.7}$$

Let T be a compact connected component of the level set $\{f_1 = c_1, \dots, f_{2n-r} = c_{2n-r}\}$ and $df_1 \wedge \dots \wedge df_{2n-r} \neq 0$ on T. Then T is diffeomorphic to a pre-isotropic $r+1$-dimensional torus $\mathbb{T}^{r+1}$. There exist a neighborhood U of T and a diffeomorphism